Cubical (ω, p)-categories

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Abstract

In this article we introduce the notion of cubical (ω, p)-categories, for $p \in \mathbb{N} \cup \{\omega\}$. We show that the equivalence between globular and groupoid ω-categories proven by Al-Agl, Brown and Steiner induces an equivalence between globular and cubical (ω, p)-categories for all $p \geqslant 0$. In particular we recover in a more explicit fashion the equivalence between globular and cubical groupoids proven by Brown and Higgins.

We also define the notion of (ω, p)-augmented directed complexes, and show that Steiner's adjunction between augmented directed complexes and globular ω-categories induces adjunctions between (ω, p)-augmented directed complexes and both globular and cubical (ω, p)-categories.

Combinatorially, the difficulty lies in defining the appropriate notion of invertibility for a cell in a cubical ω-category. We investigate three such possible definitions and the relationship between them. We show that cubical $(\omega, 1)$-categories have a natural structure of symmetric cubical categories. We give an explicit description of the notions of lax, oplax and pseudo transfors between cubical categories, the latter making use of the notion of invertible cell defined previously.

Contents

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1 Cubical categories

In this Section we recall the notion of ω-cubical categories (with connections) and the following functors

$$\begin{tikzcd} \omega\text{-}\mathbf{Cat} \arrow[rr, phantom, "\cong"] & & \omega\text{-}\mathbf{CubCat} \arrow[ll, bend right, "\lambda"'] \\ \end{tikzcd}$$

$$\begin{tikzcd} \omega\text{-}\mathbf{Cat} \arrow[rr, bend right, "\gamma"'] & & \omega\text{-}\mathbf{CubCat} \end{tikzcd}$$

defined in [1] that form an equivalence between the category of cubical ω-categories and that of globular ω-categories.

While our description of the functor λ matches exactly the description given in [1], we rephrase slightly the definition of γ. Our construction consists in defining a co-*cubical ω-category* object in $\omega\text{-}\mathbf{Cat}$ (that is a cubical ω-category object in $\omega\text{-}\mathbf{Cat}^{\mathrm{op}}$), n order to define γ as a nerve functor. The starting point of this construction consists in describing the *standard globular ω-category of the n-cube* (denoted $n\text{-}\blacksquare^{\mathbf{G}}$ in this paper, and $M(I^n)$ in [1]). Here we use the closed monoidal structure on $\omega\text{-}\mathbf{Cat}$ to construct these categories, but one could equivalently define them as in [1] using directed complexes [28], or using augmented directed complexes [29].

1.1 Cubical sets

Definition 1.1.1. We denote by $n\text{-}\mathbf{Cat}$ the category of strict globular n-categories (with $n \in \mathbb{N} \cup \{\omega\}$). We implicitly consider all globular n-categories (with $n \in \mathbb{N}$) to be globular ω-categories whose only cells in dimension higher than n are identities. Let $\mathcal{C}$ be a globular ω-category and $n \geqslant 0$. We denote by $\mathcal{C}_n$ the set of n-cells of $\mathcal{C}$. For $f \in \mathcal{C}_n$, and $0 \leqslant k < n$, we denote by $\mathrm{s}_k(f) \in \mathcal{C}_k$ (resp. $\mathrm{t}_k(f)$) the k-dimensional source (resp. target) of f, and we simply write $\mathrm{s}(f)$ (resp. $\mathrm{t}(f)$) for $\mathrm{s}_{n-1}(f)$ (resp. $\mathrm{t}_{n-1}(f)$). For $f, g \in \mathcal{C}_n$ such that $\mathrm{t}_k(f) = \mathrm{s}_k(g)$ we write $f \bullet_k g$ their composite. For $f \in \mathcal{C}_n$ we write 1_f the identity of f. Finally for $x, y \in \mathcal{C}_0$, we denote by $\mathcal{C}(x, y)$ the globular ω-category of arrows between them.

We say that an n-cell $f \in \mathcal{C}_n$ is *invertible* if it is invertible for the composition $\bullet_{n-1}$, that is if there exists an n-cell $g \in \mathcal{C}_n$ such that $f \bullet_{n-1} g = 1_{\mathrm{s}(f)}$ and $g \bullet_{n-1} f = 1_{\mathrm{t}(f)}$. For $p \geqslant 0$, a *globular (ω, p)-category* is a globular ω-category in which any n-cell is invertible, for $n > p$. In particular, a globular $(\omega, 0)$-category is just a globular ω-groupoid.

Definition 1.1.2. For every $i \in \mathbb{N}$, we define two applications $(\_)^i : \mathbb{N} \to \mathbb{N}\backslash\{i\}$ and $(\_)_i : \mathbb{N}\backslash\{i\} \to \mathbb{N}$ as follows:

$$j^i := \begin{cases} j & j < i \\ j+1 & j \geqslant i \end{cases} \qquad\qquad j_i := \begin{cases} j & j < i \\ j-1 & j > i \end{cases}$$

Finally, let i, j be distinct integers. We define application $(\_)_{i,j}$, $(\_)^{i,j}$ and $(\_)^j_i$ respectively as follows:

$$\begin{cases} \mathbb{N}\backslash\{i,j\} & \to & \mathbb{N} \\ k & \mapsto & (k_i)_{j_i} \end{cases} \qquad \begin{cases} \mathbb{N} & \to & \mathbb{N}\backslash\{i,j\} \\ k & \mapsto & (k^{i_j})^j \end{cases} \qquad \begin{cases} \mathbb{N}\backslash\{i_j\} & \to & \mathbb{N}\backslash\{j_i\} \\ k & \mapsto & (k^j)_i \end{cases}$$

Lemma 1.1.3. *The following equalities hold, for every k and every $i \neq j$:*

$$\begin{cases} k^{i,j} = k^{j,i} & \\ k_{i,j} = k_{j,i} & k \neq i, j \\ k^j_i = (k_{i_j})^{j_i} & k \neq i_j \end{cases}$$

Proof. Recall that there is at most one isomorphism between any two well-ordered sets. Here $(\_)^{i,j}$ and $(\_)^{j,i}$ are both isomorphism from $\mathbb{N}$ to $\mathbb{N}\backslash\{i,j\}$, hence they are equal. The same reasoning proves the other two equalities. $\square$

The series of Definitions 1.1.4, 1.1.6 and 1.2.1 is exactly the same as in [1], except that we make use of the notations introduced in Definition 1.1.2.

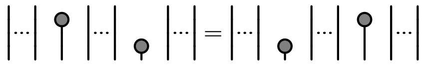
Definition 1.1.4. A pre-cubical set is a series of sets C_n (for $n \geqslant 0$) together with applications (called *faces operations*) $\partial_i^\alpha : C_n \to C_{n-1}$, for $\alpha = \pm$ and $1 \leqslant i \leqslant n$, satisfying

$$\partial^\alpha_{i_j}\partial^\beta_j = \partial^\beta_{j_i}\partial^\beta_i \tag{1.1}$$

A morphism of pre-cubical sets is is a family of applications $F_n : C_n \to D_n$ commuting with the faces operations.

Example 1.1.5. Following work of Grandis and Mauri [18], pre-cubical sets can be seen as presheaves over the free PRO generated by cells : $0 \to 1$ and : $0 \to 1$. Then the applications $\partial_i^- : C_n \to C_{n-1}$ and $\partial_i^+ : C_n \to C_{n-1}$ correspond respectively to the following cells, with $i-1$ strings on the left and $n-i$ on the right:

Equation (1.1) corresponds to equations of the following form, replacing the occurrences of either by or depending on α and β:



Note finally that reading an expression $\partial_i^\alpha \dots \partial_j^\beta$ from left to right corresponds to reading a string diagram in the PRO from top to bottom.

Definition 1.1.6. A cubical set (with connections) is given by:

- For all $n \in \mathbb{N}$, a set C_n,
- For all $n \in \mathbb{N}^*$, all $1 \leqslant i \leqslant n$ and all $\alpha \in \{+,-\}$, applications $\partial_i^\alpha : C_n \to C_{n-1}$.
- For all $n \in \mathbb{N}$ and all $1 \leqslant i \leqslant n+1$, applications $\epsilon_i : C_n \to C_{n+1}$.
- For all $n \in \mathbb{N}^*$, all $1 \leqslant i \leqslant n$ and all $\alpha \in \{+,-\}$, applications $\Gamma_i^\alpha : C_n \to C_{n+1}$.

This data must moreover verify the following axioms:

$$\partial_i^\alpha \epsilon_j = \begin{cases} \epsilon_{j_i}\partial^\alpha_{i_j} & i \neq j \\ \mathrm{id}_{C_n} & i = j \end{cases} \tag{1.2}$$

$$\partial_i^\alpha \Gamma_j^\beta = \begin{cases} \Gamma^\beta_{j_i}\partial_{i_j} & i \neq j, j+1 \\ \mathrm{id}_{C_n} & i = j, j+1 \text{ and } \alpha = \beta \\ \epsilon_j \partial_j^\alpha & i = j, j+1 \text{ and } \alpha = -\beta \end{cases} \tag{1.3}$$

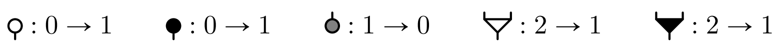
$$\epsilon_i \epsilon_{j^i} = \epsilon_j \epsilon_{i^j} \tag{1.4}$$

$$\Gamma^\alpha_{i^j}\Gamma^\beta_j = \begin{cases} \Gamma^\beta_{j^i}\Gamma^\alpha_i & i \neq j \\ \Gamma^\alpha_i\Gamma^\alpha_i & i = j \text{ and } \alpha = \beta \end{cases} \tag{1.5}$$

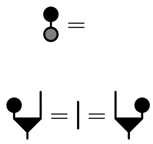
$$\Gamma_i^\alpha \epsilon_j = \begin{cases} \epsilon_{j^i}\Gamma^\alpha_{i_j} & i \neq j \\ \epsilon_i\epsilon_i & i = j \end{cases} \tag{1.6}$$

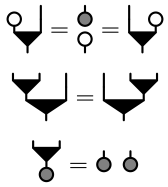
Example 1.1.7. Following once again [18], cubical sets with connections can be seen as presheaves over the following PRO, denoted by $\mathbb{J}$ and called the *intermediate cubical site* in [18]:

- The generators are the cells :



- They satisfy the following relations :





Then the applications $\Gamma_i^- : C_n \to C_{n+1}$, $\Gamma_i^+ : C_n \to C_{n+1}$ and ϵ_i correspond respectively to composites of the form $|\cdots|$ $|\cdots|$, $|\cdots|$ $|\cdots|$ and $|\cdots|$ $|\cdots|$, with the appropriate number of strings on each side.

1.2 Cubical ω-categories

Definition 1.2.1. A cubical ω-category is given by a cubical set $\mathbf{C}$, equipped with, for all $n \in \mathbb{N}^*$ and all $1 \leqslant i \leqslant n$, a partial application $\star_i$ from $\mathbf{C}_n \times \mathbf{C}_n$ to $\mathbf{C}_n$ defined exactly for any cells A,B such that $\partial_i^+ A = \partial_i^- B$. This data must moreover satisfy the following axioms:

$$(A \star_i B) \star_j (C \star_i D) = (A \star_j C) \star_i (B \star_j D) \quad (1.7)$$

$$A \star_i (B \star_i C) = (A \star_i B) \star_i C \quad (1.8)$$

$$\epsilon_i(A \star_j B) = \epsilon_i^\alpha A \star_{j^i} \epsilon_i^\alpha B \quad (1.9)$$

$$A \star_i \epsilon_i \partial_i^+ A = \epsilon_i \partial_i^- A \star_i A = A \quad (1.10)$$

$$\Gamma_i^+ A \star_i \Gamma_i^- A = \epsilon_{i+1} A \quad (1.11)$$

$$\Gamma_i^+ A \star_{i+1} \Gamma_i^- A = \epsilon_i A \quad (1.12)$$

$$\partial_i^\alpha (A \star_j B) = \begin{cases} \partial_i^\alpha A \star_{j_i} \partial_i^\alpha B & i \neq j \\ \partial_i^- A & i = j \text{ and } \alpha = - \\ \partial_i^+ B & i = j \text{ and } \alpha = + \end{cases} \quad (1.13)$$

$$\Gamma_i^\alpha (A \star_j B) = \begin{cases} \Gamma_i^\alpha A \star_{j^i} \Gamma_i^\alpha B & i \neq j \\ \begin{array}{|c|c|} \hline \Gamma_i^- A & \epsilon_{i+1} B \\ \hline \epsilon_i B & \Gamma_i^- B \\ \hline \end{array} \quad \begin{array}{l} \xrightarrow{\quad i} \\ \downarrow_{i+1} \end{array} & i = j \text{ and } \alpha = - \\ \begin{array}{|c|c|} \hline \Gamma_i^+ A & \epsilon_i A \\ \hline \epsilon_{i+1} A & \Gamma_i^+ B \\ \hline \end{array} \quad \begin{array}{l} \xrightarrow{\quad i} \\ \downarrow_{i+1} \end{array} & i = j \text{ and } \alpha = + \end{cases} \quad (1.14)$$

where in the last relation we denote by $\begin{array}{|c|c|} \hline A & B \\ \hline C & D \\ \hline \end{array} \quad \begin{array}{l} \xrightarrow{\quad i} \\ \downarrow_{j} \end{array}$ the composite $(A \star_i B) \star_j (C \star_i D)$ (which is made possible by relation (1.7)). We denote by ω - **CubCat** the category of cubical ω-category.

Definition 1.2.2. Let $\mathbf{C}$ be a cubical ω-category. For any $n > 0$, we define operations ψ_i, Ψ_r, Φ_m : $\mathbf{C}_n \to \mathbf{C}_n$, with $1 \leqslant i \leqslant n-1$, $1 \leqslant r \leqslant n$ and $0 \leqslant m \leqslant n$ as follows:

$$\begin{array}{rcl} \psi_i A & = & \Gamma_i^+ \partial_{i+1}^- A \star_{i+1} A \star_{i+1} \Gamma_i^- \partial_{i+1}^+ A \\ \Psi_r A & = & \psi_{r-1} \cdots \psi_1 A \\ \Phi_m A & = & \Psi_1 \cdots \Psi_m A \end{array}$$

Definition 1.2.3. Let $\mathbf{C}$ be a cubical ω-category, and $A \in \mathbf{C}_n$. We say that A is a *thin* cell if $\psi_1 \ldots \psi_{n-1} A \in \mathbf{Im}\, \epsilon_1$.

Definition 1.2.4. Let $n \in \mathbb{N}$. There is a truncation functor $\mathrm{tr}_n : (n+1)$ - **CubCat** $\to n$ - **CubCat**. This functor admits both a left and a right adjoint (see [20] for an explicit description of both functors).

$$
\begin{tikzcd}
(n+1)\text{-}\mathbf{CubCat} \arrow[rr, "\mathrm{tr}_n" description] \arrow[rr, phantom, bend left=20, "\perp"] \arrow[rr, phantom, bend right=20, "\perp"] & & n\text{-}\mathbf{CubCat} \arrow[ll, bend right=40, "\boxslash"'] \arrow[ll, bend left=40, "\square"]
\end{tikzcd}
$$

For $\mathbf{C} \in n\text{-}\mathbf{CubCat}$, the $(n+1)$-category $\square\mathbf{C}$ coincides with $\mathbf{C}$ up to dimension n, and the rest of the structure is defined as follows:

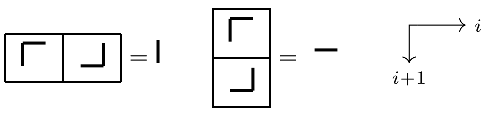
- The set of $(n+1)$-cells of $\square\mathbf{C}$ is the set of all families $(A_i^\alpha) \in \mathbf{C}_n$ (with $1 \leqslant i \leqslant n+1$ and $\alpha = \pm$) such that:
$$\partial_{i_j}^\alpha A_j^\beta = \partial_{j_i}^\beta A_i^\alpha.$$
- For $A \in (\square\mathbf{C})_{n+1}$, $\partial_i^\alpha A = A_i^\alpha$.
- For $A \in \mathbf{C}_n$, the families $\epsilon_i A \in (\square\mathbf{C})_{n+1}$ and $\Gamma_i^\alpha A \in (\square\mathbf{C})_{n+1}$ are defined by:
$$(\epsilon_i A)_j^\beta = \begin{cases} A & j = i \\ \epsilon_{i_j}\partial_{j_i}^\beta A & j \neq i \end{cases} \qquad (\Gamma_i^\alpha)_j^\beta = \begin{cases} A & j = i, i+1 \text{ and } \beta = \alpha \\ \epsilon_i \partial_i^\beta A & j = i, i+1 \text{ and } \beta = -\alpha \\ \Gamma_{i_j}^\alpha \partial_{j_i} A & j \neq i, i+1 \end{cases}$$
- For $A, B \in (\square\mathbf{C})_{n+1}$ such that $A_i^+ = B_i^-$, the family $A \star_i B \in (\square\mathbf{C})_{n+1}$ is defined by:
$$(A \star_i B)_j^\alpha = \begin{cases} A_i^- & j = i \text{ and } \alpha = - \\ B_i^+ & j = i \text{ and } \alpha = + \\ A_j^\alpha \star_{i_j} B_j^\alpha & j \neq i \end{cases}$$

Let $\mathbf{C}$ be a cubical $(n+1)$-category. The unit of the adjunction $\mathrm{tr} \dashv \square$ induces a morphism of cubical $(n+1)$-categories $\boldsymbol{\partial} : \mathbf{C} \to \square\,\mathrm{tr}\,\mathbf{C}$. This functor associates, to any $A \in \mathbf{C}_{n+1}$ the family $\boldsymbol{\partial}A := (\partial_i^\alpha A)$. We call $\boldsymbol{\partial}A$ the *shell* of A.

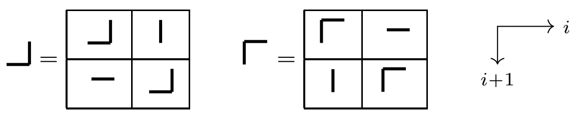
More generally, if $\mathbf{C}$ is a cubical ω-category, we denote by $\square_n\mathbf{C}$ the $(n+1)$-category $\square\,\mathrm{tr}_n\,\mathbf{C}$, and for any $A \in \mathbf{C}_{n+1}$, by $\boldsymbol{\partial}A$ the cell $\boldsymbol{\partial}\,\mathrm{tr}_{n+1} A \in \square_n\mathbf{C}$.

Theorem 1.2.5 (Proposition 2.1 and Theorem 2.8 from [20])**.** *Let $\mathbf{C}$ be a cubical category. Thin cells of $\mathbf{C}$ are exactly the composites of cells of the form $\epsilon_i f$ and $\Gamma_i^\alpha f$. Moreover, if two thin cells have the same shell, then they are equal.*

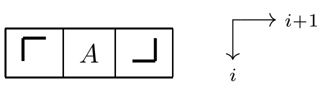
Notation 1.2.6. As a consequence, when writing thin cells in 2-dimensional compositions (as in Equation (1.14) for example), we make use of the notation already used in [1] and [20]: a thin cell A is replaced by a string diagram linking the non-thin faces of A. For example $\Gamma_i^+ A$ and $\Gamma_i^- A$ will respectively be represented by the symbols ┌ and ┘, and the cells $\epsilon_i A$ by the symbol ─ or |. Following this convention, Equations (1.11) and (1.12) can be represented by the following string diagrams:



And the last two cases of Equation (1.14) become respectively:



Finally, for any $A \in \mathcal{C}_n$, $\psi_i A$ is the following composite:

$\psi_i A =$ 

1.3 Equivalence between cubical and globular ω-categories

The functor $\gamma : \omega\text{-}\mathbf{CubCat} \to \omega\text{-}\mathbf{Cat}$ was described in [1] as follows.

Proposition 1.3.1. *Let* $\mathbf{C}$ *be a cubical category. The following assignment defines a globular ω-category* $\gamma\mathbf{C}$ *:*

- *The n-cells of $\gamma\mathcal{C}$ is the set* $\Phi_n(\mathbf{C}_n)$*,*
- *For all* $A \in \gamma\mathbf{C}_n$*,* $1_A := \epsilon_1 A$*,*
- *For all* $A \in \gamma\mathbf{C}_n$*,* $\mathrm{s}(A) := \partial_1^- A$*,*
- *For all* $A \in \gamma\mathbf{C}_n$*,* $\mathrm{t}(A) := \partial_1^+ A$*,*
- *For all* $A, B \in \gamma\mathbf{C}_n$ *and* $0 \leqslant k < n$*,* $A \bullet_k B := A \star_{n-k} B$*.*

To define the functor $\lambda : \omega\text{-}\mathbf{Cat} \to \omega\text{-}\mathbf{CubCat}$, we start by constructing a co-cubical ω-category object in $\omega\text{-}\mathbf{Cat}$. This is a reformulation of [1].

Definition 1.3.2. Let $\mathbb{I}$ be the category with two 0-cells $(-)$ and $(+)$ and one non-identity 1-cell (θ):

$$(\theta) : (-) \to (+)$$

We denote by $n\text{-}\blacksquare^{\mathbf{G}}$, and call the *$n$-cube category* the globular ω-category $\mathbb{I}^{\otimes n}$, where $\otimes$ is the Crans-Gray tensor product, which equips $\omega\text{-}\mathbf{Cat}$ with a closed monoidal structure.

Example 1.3.3. For example $2\text{-}\blacksquare^{\mathbf{G}}$ is the free 2-category with four 0-cells, four generating 1-cells and one generating 2-cell, with source and targets given by the following diagram:

$$\begin{tikzcd}
(--) \arrow[r, "(\theta -)"] \arrow[d, "(-\theta)"'] & (+-) \arrow[d, "(+\theta)"] \arrow[dl, Rightarrow, shorten=12pt, "A" description] \\
(-+) \arrow[r, "(\theta +)"'] & (++)
\end{tikzcd}$$

Definition 1.3.4. For $\alpha = \pm$, we denote by $\check{\partial}^\alpha : \top \to \mathbb{I}$ the functor sending the (unique) 0-dimensional cell of $\top$ to (α), where $\top$ denotes the terminal category.

For any $n \geqslant 0$, any $1 \leqslant i \leqslant n$ and any $\alpha = \pm$, we denote by $\check{\partial}_i^\alpha : n\text{-}\blacksquare^{\mathbf{G}} \to (n+1)\text{-}\blacksquare^{\mathbf{G}}$ the functor $\mathbb{I}^{i-1} \otimes \check{\partial}^\alpha \otimes \mathbb{I}^{n-i}$.

Definition 1.3.5. We denote by $\check{\epsilon} : 1\text{-}\blacksquare^{\mathbf{G}} \to 0\text{-}\blacksquare^{\mathbf{G}}$ the (unique) functor from $\mathbb{I}$ to $\top$.

For any $n > 0$ and any $1 \leqslant i \leqslant n$, we denote by $\check{\epsilon}_i : (n-1)\text{-}\blacksquare^{\mathbf{G}} \to n\text{-}\blacksquare^{\mathbf{G}}$ the functor $\mathbb{I}^{\otimes(i-1)} \otimes \check{\epsilon} \otimes \mathbb{I}^{\otimes(n-i)}$.

Definition 1.3.6. For $\alpha = \pm$, let $\check{\Gamma}^\alpha : 2\text{-}\blacksquare^{\mathbf{G}} \to 1\text{-}\blacksquare^{\mathbf{G}}$ be the functor defined as follows, where $\beta = -\alpha$:

$$\begin{cases} \check{\Gamma}^\alpha(\alpha\alpha) = (\alpha) \\ \check{\Gamma}^\alpha(\alpha\beta) = (\beta) \\ \check{\Gamma}^\alpha(\beta\alpha) = (\beta) \\ \check{\Gamma}^\alpha(\beta\beta) = (\beta) \end{cases} \qquad \begin{cases} \check{\Gamma}^\alpha(\theta\alpha) = (\theta) \\ \check{\Gamma}^\alpha(\theta\beta) = 1_{(\beta)} \\ \check{\Gamma}^\alpha(\alpha\theta) = (\theta) \\ \check{\Gamma}^\alpha(\beta\theta) = 1_{(\beta)} \end{cases} \qquad \check{\Gamma}^\alpha(\theta\theta) = 1_{(\theta)}$$

For any $n > 0$ any $1 \leqslant i \leqslant n$ and any $\alpha = \pm$, we denote by $\check{\Gamma}_i^\alpha : n\text{-}\blacksquare^{\mathbf{G}} \to (n+1)\text{-}\blacksquare^{\mathbf{G}}$ the functor $\mathbb{I}^{\otimes(i-1)} \otimes \check{\Gamma}^\alpha \otimes \mathbb{I}^{\otimes(n-i)}$.

Definition 1.3.7. We denote by $\mathbf{Rect^G}$ the following coproduct in $\omega\text{-}\mathbf{Cat}$:

$$\begin{tikzcd} 0\text{-}\blacksquare^{\mathbf{G}} \arrow[r, "\check{\partial}^{+}"] \arrow[d, "\check{\partial}^{-}"'] \arrow[dr, phantom, "\ulcorner"] & 1\text{-}\blacksquare^{\mathbf{G}} \arrow[d] \\ 1\text{-}\blacksquare^{\mathbf{G}} \arrow[r] & \mathbf{Rect^G} \end{tikzcd} \tag{1.15}$$

Explicitly, the 0-cells of $\mathbf{Rect^G}$ are elements (α_j), where $\alpha = \pm$ and $i = 1, 2$, with the identification $(+_1) = (-_2)$. The 1-cells of $\mathbf{Rect}_0^{\mathbf{ADC}}$ are freely generated by $(\theta_i) : (-_i) \to (+_i)$, for $i = 1, 2$.

For every $n > 0$ and every $1 \leqslant i \leqslant n$, let $(n, i)\text{-}\mathbf{Rect^G}$ be the cubical ω-category:

$$(n,i)\text{-}\mathbf{Rect^G} := \mathbb{I}^{(i-1)} \otimes \mathbf{Rect^G} \otimes \mathbb{I}^{(n-i)}$$

Remark 1.3.8. Since the monoidal structure on $\omega\text{-}\mathbf{Cat}$ is biclosed [1], $(n, i)\text{-}\mathbf{Rect^G}$ is the colimit of the following diagram:

$$\begin{tikzcd} (n-1)\text{-}\blacksquare^{\mathbf{G}} \arrow[r, "\check{\partial}_i^{+}"] \arrow[d, "\check{\partial}_i^{-}"'] \arrow[dr, phantom, "\ulcorner"] & n\text{-}\blacksquare^{\mathbf{G}} \arrow[d] \\ n\text{-}\blacksquare^{\mathbf{G}} \arrow[r] & (n,i)\text{-}\mathbf{Rect^G} \end{tikzcd} \tag{1.16}$$

Definition 1.3.9. We denote by $\check{\star} : 1\text{-}\blacksquare^{\mathbf{G}} \to \mathbf{Rect^G}$ the following functor:

$$\begin{cases} \check{\star}(-) = (-_1) \\ \check{\star}(+) = (+_2) \end{cases} \qquad \check{\star}(\theta) = (\theta_1) \bullet_0 (\theta_2)$$

For any $n > 0$ and any $1 \leqslant i \leqslant n$, we denote by $\check{\star}_i : n\text{-}\blacksquare^{\mathbf{G}} \to (n, i)\text{-}\mathbf{Rect^G}$ the functor $\mathbb{I}^{\otimes(i-1)} \otimes \check{\star} \otimes \mathbb{I}^{\otimes(n-i)}$.

This result is a reformulation of Section 2 of [1]:

Proposition 1.3.10. *The objects $n\text{-}\blacksquare^{\mathbf{G}}$ equipped with the applications $\check{\partial}_i^{\alpha}$, $\check{\epsilon}_i$, $\check{\Gamma}_i^{\alpha}$ and $\check{\star}_i$ form a co-*cubical ω-category *object in the category $\omega\text{-}\mathbf{Cat}$.*

Consequently, for $\mathcal{C}$ a globular ω-category, the family $(\lambda\mathcal{C})_n = \omega\text{-}\mathbf{Cat}(n\text{-}\blacksquare^{\mathbf{G}}, \mathcal{C})$ comes equipped with a structure of cubical ω-category, that we denote by $\lambda\mathcal{C}$. This defines a functor $\lambda : \omega\text{-}\mathbf{Cat} \to \omega\text{-}\mathbf{CubCat}$.

Finally, the main result of [1] is the following:

Theorem 1.3.11. *The following functors form an equivalence of Categories:*

$$\begin{tikzcd} \omega\text{-}\mathbf{Cat} \arrow[rr, bend right, "\gamma"'] \arrow[rr, phantom, "\cong"] & & \omega\text{-}\mathbf{CubCat} \arrow[ll, bend right, "\lambda"'] \end{tikzcd}$$

2 Invertible cells in cubical ω-categories

In this Section, we investigate three notions of invertibility in cubical ω-categories. We start by defining in Section 2.1, both the notion of R_i-invertibility, which is a direct cubical analogue of the usual notion of invertibilty with respect to a binary composition, and the notion of (plain) invertibility, which is specific to cubical ω-categories. Then in Section 2.3, we define a notion of T_i-invertibility, a variant of the notion of R_i-invertibility using a kind of diagonal composition.

2.1 R_i-invertibility

We start by proving a number of useful Lemmas about the notion of R_i-invertibility. We then proceed to give the definition of (plain) invertibility in Definition 2.2.1. The rest of the Section is then used to prove a characterisation of R_i-invertibility in terms of invertibility, which is achieved in Proposition 2.2.2.

Definition 2.1.1. Let $\mathbf{C}$ be a cubical ω-category, and $1 \leqslant k \leqslant n$ be integers. We say that a cell $A \in \mathbf{C}_n$ is R_k*-invertible* if there exists $B \in \mathbf{C}_n$ such that $A \star_k B = \epsilon_k \partial_k^- A$ and $B \star_k A = \epsilon_k \partial_k^+ A$. We call B the R_k-inverse of A, and we write $R_k A$ for B.

In particular, we say that $A \in \mathbf{C}_n$ has an R_k-invertible shell if $\boldsymbol{\partial} A$ is R_k-invertible in $\square_n \mathbf{C}$.

Lemma 2.1.2. *Let $\mathbf{C}$ be a cubical n-category, and $A \in (\square\mathbf{C})_{n+1}$. Then A is R_i-invertible if and only if for all $j \neq i$, A_j^α is R_{i_j}-invertible, and:*

$$\partial_i^\alpha R_k A = \begin{cases} \partial_k^{-\alpha} A & i = k \\ R_{k_i} \partial_i^\alpha A & i \neq k \end{cases}$$

In particular, for $\mathbf{C}$ a cubical ω-categories, a cell $A \in \mathbf{C}_n$ has an R_i-invertible shell if and only if $\partial_j^\alpha A$ is R_{i_j}-invertible for any $j \neq i$.

Proof. Let B be the R_k-inverse of A, and $i \neq k$. We have:

$$A_i^\alpha \star_{k_i} B_i^\alpha = (A \star_k B)_i^\alpha = \partial_i^\alpha \epsilon_k A_k^- = \epsilon_{k_i} \partial_{i^k}^\alpha A_k^- = \epsilon_{k_i} \partial_{k_{i^k}}^- A_i^\alpha = \epsilon_{k_i} \partial_{k_i}^- A_i^\alpha,$$
$$B_i^\alpha \star_{k_i} A_i^\alpha = (B \star_k A)_i^\alpha = \partial_i^\alpha \epsilon_k A_k^+ = \epsilon_{k_i} \partial_{i^k}^\alpha A_k^+ = \epsilon_{k_i} \partial_{k_{i^k}}^+ A_i^\alpha = \epsilon_{k_i} \partial_{k_i}^+ A_i^\alpha.$$

Thus B_i^α is the k_i-inverse of A_i^α, that is $\partial_i^\alpha R_k A = R_{k_i} \partial_i^\alpha A$.

Moreover, for the composite $A \star_k R_k A$ (resp. $R_k A \star_k A$) to make sense we necessarily have $\partial_k^- R_k A = \partial_k^+ A$ (resp. $\partial_k^+ R_k A = \partial_k^- A$). □

The following Lemma will be useful in order to compute the R_i-inverses of thin cells.

Lemma 2.1.3. *Let $\mathbf{C}$ be a cubical ω-category, and let A be a thin cell in $\mathbf{C}_n$. We fix an integer $i \leqslant n$. If there exists a thin cell B in $\mathbf{C}_n$ such that $\partial_i^\alpha B = \partial_i^{-\alpha} A$, and for all $j \neq i$, $\partial_j^\alpha B = R_{i_j} \partial_j^\alpha A$, then A is R_i-invertible, and $B = R_i A$.*

Proof. Since $\partial_i^- B = \partial_i^+ A$, A and B are i-composable. Let us look at the cell $A \star_i B$. It is a thin cell, and it has the following shell:

$$\partial_j^\alpha (A \star_i B) = \begin{cases} \partial_i^- A = \partial_i^- \epsilon_i \partial_i^- A & j = i \text{ and } \alpha = - \\ \partial_i^+ B = \partial_i^- A = \partial_i^- \epsilon_i \partial_i^- A & j = i \text{ and } \alpha = + \\ \partial_j^\alpha A \star_{i_j} \partial_j^\alpha B = \partial_j^\alpha A \star_{i_j} R_{i_j} \partial_j^\alpha A = \epsilon_{i_j} \partial_{i_j}^- \partial_j^\alpha A = \partial_j^\alpha \epsilon_i \partial_i^- A & j \neq i \end{cases}$$

Therefore, $A \star_i B$ and $\epsilon_i \partial_i^- A$ are two thin cells that have the same shell. By Theorem 1.2.5, they are equal. The same computation with $B \star_i A$ leads to the equality $B \star_i A = \epsilon_i \partial_i^+ A$. Finally, A is R_i-invertible, and $R_i A = B$. □

Lemma 2.1.4. *Let $\mathbf{C}$ be a cubical ω-category, and fix $A, B \in \mathbf{C}_n$ and $1 \leqslant k \leqslant n$.*

- *For any $i \leqslant n$, if A, B are R_k-invertible and i-composable, then $A \star_i B$ is R_k-invertible, and:*

$$R_k (A \star_i B) = \begin{cases} R_k A \star_i R_k B & i \neq k \\ R_k B \star_k R_k A & i = k \end{cases} \tag{2.1}$$

- *For any $i \leqslant n+1$, $\epsilon_i A$ is R_i-invertible and $R_i \epsilon_i A = \epsilon_i A$. Moreover if A is R_k-invertible then $\epsilon_i A$ is also R_{k^i} invertible, with*
$$R_{k^i} \epsilon_i A = \epsilon_i R_k A \tag{2.2}$$
- *For any $i \neq k$ and $\alpha = \pm$, if A is R_k-invertible, then $\Gamma_i^\alpha A$ is R_{k^i} invertible, and $\Gamma_k^\alpha A$ is both R_k and R_{k+1}-invertible, and:*
$$R_{k^i} \Gamma_i^\alpha A = \Gamma_i^\alpha R_k A \tag{2.3}$$
$$R_k \Gamma_k^\alpha A = \begin{cases} \epsilon_{k+1} R_k A \star_{k+1} \Gamma_k^+ A & \alpha = - \\ \Gamma_k^- A \star_k \epsilon_{k+1} R_k A & \alpha = + \end{cases} \qquad R_{k+1} \Gamma_k^\alpha A = \begin{cases} \epsilon_k R_k A \star_{k+1} \Gamma_k^+ A & \alpha = - \\ \Gamma_k^- A \star_{k+1} \epsilon_k R_k A & \alpha = + \end{cases} \tag{2.4}$$

Proof. Suppose A and B are k-invertible, and let $i \leqslant n$. If $i \neq k$, Then we have:
$$(R_k A \star_i R_k B) \star_k (A \star_i B) = (R_k A \star_k A) \star_i (R_k B \star_k B) = \epsilon_k \partial_k^+ A \star_i \epsilon_k \partial_k^+ B = \epsilon_k \partial_k^+ (A \star_i B)$$
$$(A \star_i B) \star_k (R_k A \star_i R_k B) = (A \star_k R_k A) \star_i (B \star_k R_k B) = \epsilon_k \partial_k^- A \star_i \epsilon_k \partial_k^- B = \epsilon_k \partial_k^- (A \star_i B).$$
Thus $A \star_i B$ is R_k-invertible and $R_k(A \star_i B) = R_k A \star_i R_k B$. Suppose now that $i = k$. Then we have:
$$R_k B \star_k R_k A \star_k A \star_k B = \epsilon_k \partial_k^+ B = \epsilon_k \partial_k^+ (A \star_k B)$$
$$A \star_k B \star_k R_k B \star_k R_k A = \epsilon_k \partial_k^- A = \epsilon_k \partial_k^- (A \star_k B).$$
So $A \star_k B$ is R_k-invertible, and $R_k(A \star_k B) = R_k B \star_k R_k A$.

Suppose $i \neq k$. Then we have:
$$\Gamma_i^\alpha A \star_{k^i} \Gamma_i^\alpha R_k A = \Gamma_i^\alpha (A \star_k R_k A) = \Gamma_i^\alpha \epsilon_k \partial_k^- A = \epsilon_{k^i} \Gamma_{i_k}^\alpha \partial_k^- A = \epsilon_{k^i} \partial_{k^i}^- \Gamma_i^\alpha A$$
$$\Gamma_i^\alpha R_k A \star_{k^i} \Gamma_i^\alpha A = \Gamma_i^\alpha (R_k A \star_k A) = \Gamma_i^\alpha \epsilon_k \partial_k^+ A = \epsilon_{k^i} \Gamma_{i_k}^\alpha \partial_k^+ A = \epsilon_{k^i} \partial_{k^i}^+ \Gamma_i^\alpha A$$
Thus $\Gamma_i^\alpha A$ is R_{k^i}-invertible, and $R_{k^i} \Gamma_i^\alpha A = \Gamma_i^\alpha R_k A$.

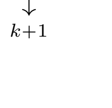
Suppose now $i = k$, and $\alpha = -$. In order to show that $R_k \Gamma_k^- A = \epsilon_{k+1} R_k A \star_{k+1} \Gamma_k^+ A$, we are going to use Lemma 2.1.3. Note first that both $\Gamma_k^- A$ and $\epsilon_{k+1} R_k A \star_{k+1} \Gamma_k^+ A$ are thin, so we only need to check the hypothesis about the shell of $\epsilon_{k+1} R_k A \star_{k+1} \Gamma_k^+ A$. Note that the hypotheses on directions k and $k+1$ are always satisfied:
$$\partial_j^\alpha (\epsilon_{k+1} R_k A \star_k \Gamma_k^+ A) = \begin{cases} \epsilon_k \partial_k^- R_k A = \epsilon_k \partial_k^+ A = \partial_k^+ \Gamma_k^- A & j = k \text{ and } \alpha = - \\ \partial_k^+ \Gamma_k^+ A = A = \partial_k^- \Gamma_k^- A & j = k \text{ and } \alpha = + \\ R_k A \star_k \partial_{k+1}^- \Gamma_k^+ A = R_k A \star_k \epsilon_k \partial_k^- A = R_k A = R_k \partial_k^- \Gamma_k^- A & j = k+1 \text{ and } \alpha = - \\ R_k A \star_k \partial_{k+1}^+ \Gamma_k^+ A = R_k A \star_k A = \epsilon_k \partial_k^+ A = R_k \partial_k^+ \Gamma_k^- A & j = k+1 \text{ and } \alpha = + \end{cases}$$

As for the remaining directions, we reason by induction on n, the dimension of A. The case where $n = 1$ (and thus $k = 1$), there is no other direction to check and so $R_1 \Gamma_1^- = \epsilon_2 R_1 A \star_2 \Gamma_1^+ A$. Suppose now $n > 1$, and let $j \leqslant n+1$, with $j \neq k, k+1$. Then we have the following equalities (where the fourth one uses the induction hypothesis):
$$\begin{aligned} \partial_j^\alpha (\epsilon_{k+1} R_k A \star_k \Gamma_k^+ A) &= \partial_j^\alpha \epsilon_{k+1} R_k A \star_{k_j} \partial_j^\alpha \Gamma_k^+ A \\ &= \epsilon_{(k+1)_j} \partial_{j_{k+1}}^\alpha R_k A \star_{k_j} \Gamma_{k_j}^+ \partial_{j_k}^\alpha A \\ &= \epsilon_{k_j+1} R_{k_j} \partial_{j_k}^\alpha A \star_{k_j} \Gamma_{k_j}^+ \partial_{j_k}^\alpha A \\ &= R_{k_j} \Gamma_{k_j}^- \partial_{j_k}^\alpha A \\ &= R_{k_j} \partial_j^\alpha \Gamma_k^- A \end{aligned}$$
Thus by Lemma 2.1.3, $\Gamma_k^- A$ is R_k-invertible, and $R_k \Gamma_k^- A = \epsilon_{k+1} R_k A \star_{k+1} \Gamma_k^+ A$.

The proofs of the remaining three cases ($i = k$ with $\alpha = +$, and $i = k+1$ with $\alpha = \pm$) are similar. $\square$

Remark 2.1.5. Note that Lemma 2.1.4 shows in particular that, if A is R_k-invertible, then $R_{k^i} \Gamma_i^\alpha$, $R_k \Gamma_k^\alpha A$ and $R_{k+1} \Gamma_k^\alpha A$ are thin. In particular, applying the Notation defined in 1.2.6, we get the equations:

$$R_k \ulcorner = \urcorner \qquad R_{k+1} \ulcorner = \llcorner \qquad R_k \lrcorner = \llcorner \qquad R_{k+1} \lrcorner = \urcorner$$



Remark 2.1.6. Let $\mathbf{C}$ be a cubical n-category and $A \in (\square\mathbf{C})_{n+1}$. Recall from [20] that for all $i \neq 1$, $\partial_i^\alpha \psi_1 \ldots \psi_n A \in \mathbf{Im}\,\epsilon_1$. Therefore by Lemma 2.1.2, $\psi_1 \ldots \psi_n A$ is R_1-invertible. So finally, any cell in $\square\mathbf{C}$ is invertible.

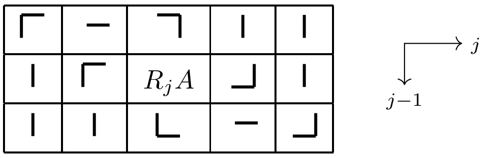
Lemma 2.1.7. *Let $\mathbf{C}$ be a cubical ω-category, and $A \in \mathbf{C}_n$. Suppose A is R_j-invertible for some $j \leqslant n$. Then :*

- *The n-cell $\psi_i A$ is R_j-invertible for any $i \neq j-1$.*
- *The n-cell $\psi_{j-1} A$ is R_{j-1}-invertible*

Proof. Suppose first $j \neq i, i+1$. Then we have $\psi_i A \star_j \psi_i R_j A = \psi_i (A \star_j R_j A) = \psi_i \epsilon_j \partial_j^- A = \epsilon_j \partial_j^- \psi_i A$, and also $\psi_i R_j A \star_j \psi_i A = \psi_i (R_j A \star_j A) = \psi_i \epsilon_j \partial_j^+ A = \epsilon_j \partial_j^+ \psi_i A$. Hence $\psi_i A$ is R_j-invertible, and $R_j \psi_i A = \psi_i R_j A$.

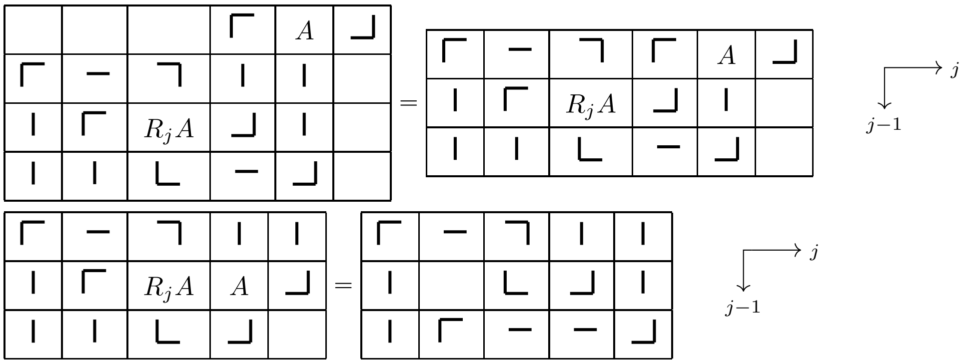
Suppose now $j = i$. Then $\psi_i A$ is a composite of R_i-invertible cells. As a consequence it is R_i-invertible.

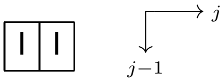
Suppose now $j = i+1$. Let B be the following composite:



Let us show that B is the R_{j-1}-inverse of $\psi_{j-1} A$:

$\psi_{j-1} A \star_{j-1} B =$





$= \epsilon_{j-1} \partial_{j-1}^- \psi_{j-1} A$

A similar computation shows that $B \star_{j-1} \psi_{j-1} A = \epsilon_{j-1} \partial_{j-1}^+ \psi_{j-1} A$ and thus $\psi_{j-1} A$ is R_{j-1}-invertible. $\square$

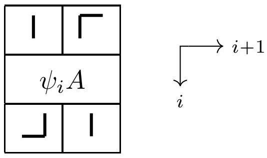
Lemma 2.1.8. *Let $\mathbf{C}$ be a cubical ω-category, and $A \in \mathbf{C}_n$ be an n-cell with an R_j-invertible shell for some $j \leqslant n$. Then:*

- *If $\psi_i A$ is R_j-invertible for some $i \neq j-1$, then A is R_j-invertible. Moreover if $R_j \psi_i A$ is thin then so is $R_j A$.*
- *If ψ_{j-1} A is R_{j-1}-invertible, then A is R_j-invertible. Moreover if $R_{j-1} \psi_{j-1} A$ is thin then so is $R_j A$.*

Proof. Suppose $\psi_i A$ is R_j-invertible, with $i \neq j$. Recall that the following composite is equal to A

<table>
<tr><td>$\epsilon_{i+1} \partial_i^- A$</td><td>$\Gamma_i^+ \partial_{i+1}^+ A$</td></tr>
<tr><td colspan="2">$\psi_i A$</td></tr>
<tr><td>$\Gamma_i^- \partial_{i+1}^- A$</td><td>$\epsilon_{i+1} \partial_i^+ A$</td></tr>
</table>

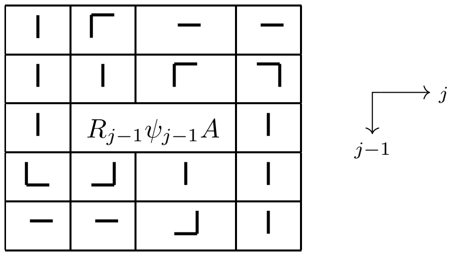
Using the string notation for thin cells, this composite can be represented as follows:



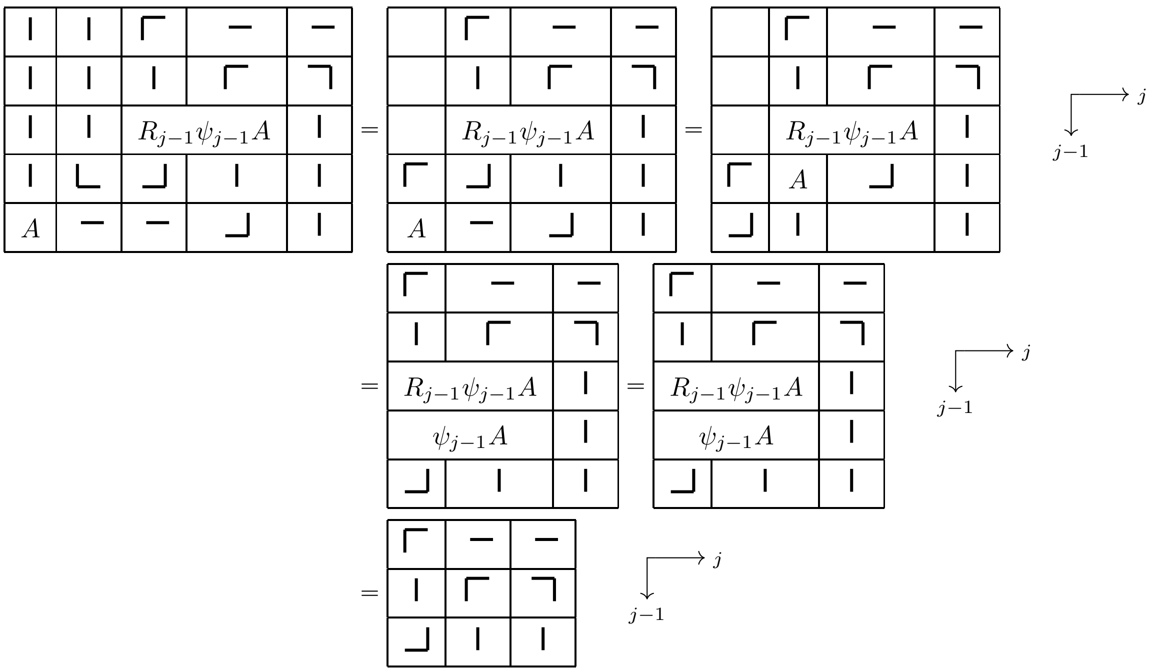
This notation is ambiguous, since it does not specify which factorisations of $\partial_i^\alpha \psi_i A$ are used. However, we use the convention that in any diagram of this form, the standard factorisations $\partial_i^- \psi_i A = \partial_i^- A \star_i \partial_{i+1}^+ A$ and $\partial_i^+ \psi_i A = \partial_{i+1}^- A \star_i \partial_i^+ A$ are used.

Since A has an R_j-invertible shell, by Lemma 2.1.4, every cell in this composite is R_j-invertible, and A is R_j-invertible. Moreover if $R_j \psi_i A$ is thin, then the explicit formulas from Lemma 2.1.4 prove that $R_j A$ is thin.

Suppose now that $\psi_{j-1} A$ is R_{j-1}-invertible. We denote by B the following composite:



We are going to show that B is the R_j-inverse of A. Notice that if $R_{j-1}\psi_{j-1} A$ is thin, then B is thin, using Lemma 2.1.4. Let us evaluate the composite $A \star_j B$:



$= \epsilon_j \partial_j^- A$

The evaluation of $B \star_j A$ is similar. □

2.2 Plain invertibility

Definition 2.2.1. We say that a cell $A \in \mathbf{C}_n$ is *invertible* if $\psi_1 \dots \psi_{n-1} A$ is R_1-invertible.

The rest of this Section is devoted to establishing the link between R_i-invertibility and (plain) invertibility. This is achieved in Proposition 2.2.2. In order to do this, we relate in Lemma 2.1.7 and 2.1.8 the R_i-invertibility of a cell A with that of $\psi_j A$.

Proposition 2.2.2. *Let* $\mathbf{C}$ *be a cubical* ω*-category,* $A \in \mathbf{C}_n$ *and* $1 \leqslant j \leqslant n$*. A cell* $A \in \mathbf{C}_n$ *is* R_j*-invertible if and only if* A *is invertible and has an* R_j*-invertible shell. Moreover if* A *is thin, then so is its* R_j*-inverse.*

Proof. Suppose first that A is R_j-invertible. Then its shell is R_j-invertible, and for all $i \geqslant j$, $\psi_i \ldots \psi_{n-1} A$ is R_j-invertible. Repeated applications of Lemma 2.3.4 show that $\psi_j \ldots \psi_{n-1} A$ is R_j-invertible. As a result (still by Lemma 2.3.4), $\psi_{j-1} \ldots \psi_{n-1} A$ is R_{j-1}-invertible. Inductively we show that for any $i \leqslant j$, $\psi_i \ldots \psi_{n-1} A$ is R_i-invertible. Finally, we get that $\psi_1 \ldots \psi_{n-1} A$ is R_1-invertible, in other words that A is invertible.

Suppose now that A is invertible and has an R_j-invertible shell. By multiple applications of Lemma 2.1.7, we get that $\psi_k \ldots \psi_{n-1} A$ has an R_j-invertible shell, for $k \geqslant j$, and an R_k-invertible one for $k \leqslant j$. Applying Lemma 2.1.8 multiple times, we get that for all $k \leqslant j$, $\psi_k \ldots \psi_{n-1} A$ is R_k-invertible, and finally that for all $k \geqslant j$, $\psi_k \ldots \psi_{n-1} A$ is R_j-invertible. In particular for $k = n$, A is R_j-invertible.

Finally if A is thin, then $\psi_1 \ldots \psi_{n-1} A \in \mathbf{Im}\, \epsilon_1$ and so $R_1 \psi_1 \ldots \psi_{n-1} A = \psi_1 \ldots \psi_{n-1} A$ is thin. Multiple applications of Lemma 2.1.8 imply that $R_j A$ is thin. ☐

Finally, the following Lemma will be useful in Proposition 2.3.5:

Lemma 2.2.3. *The composite of two invertible cells is also invertible.*

Proof. Let $1 \leqslant i \leqslant n$, and let E_i be the set of all cells $A \in \mathbf{C}_n$ such that $\psi_1 \ldots \psi_{i-1} A$ is R_1-invertible. Note first that E_i contains all R_i-invertible cells by Lemma 2.1.7 and that E_n is the set of all invertible cells. We are going to show by induction on i that E_i is closed under composition, for $1 \leqslant i \leqslant n$.

For $i = 1$, E_1 is the set of all R_1-invertible cells, which is closed under composition by Lemma 2.1.4. Suppose now $i > 1$. Take $A, B \in E_i$. We have:

$$\psi_{i-1}(A \star_j B) = \begin{cases} \psi_{i-1} A \star_j \psi_{i-1} B & j \neq i, i-1 \\ (\psi_{i-1} A \star_i \epsilon_{i-1} \partial_i^+ B) \star_{i-1} (\epsilon_{i-1} \partial_i^- A \star_i \psi_{i-1} B) & j = i-1 \\ (\epsilon_{i-1} \partial_{i-1}^- A \star_i \psi_{i-1} B) \star_{i-1} (\psi_{i-1} A \star_i \epsilon_{i-1} \partial_{i-1}^+ B) & j = i \end{cases}$$

Note that:

- Since $\psi_1 \ldots \psi_{i-1} A$ and $\psi_1 \ldots \psi_{i-1} B$ are R_1-invertible, $\psi_{i-1} A$ and $\psi_{i-1} B$ are in E_{i-1}.
- The cells $\epsilon_{i-1} \partial_k^\alpha A$ and $\epsilon_{i-1} \partial_k^\alpha B$ are R_{i-1}-invertible by Lemma 2.1.4, and therefore are in E_{i-1}.

By induction hypothesis, E_{i-1} is closed under composition, and therefore $\psi_{i-1}(A \star_j B)$ is in E_i. So $\psi_1 \ldots \psi_{i-1}(A \star_j B)$ is R_1-invertible, and so $A \star_j B$ is in E_i, which is therefore close under composition. ☐

2.3 T_i-invertiblility

The notion of T_i-invertibility is closely related to that of R_i-invertibility, as we show in Lemma 2.3.3. Consequently, a number of results from the previous Section have analogues in terms of T_i-invertibility. In particular, the characterisation of T_i-invertibility in terms of invertibility given in Proposition 2.3.5 is the direct analogue of Proposition 2.2.2.

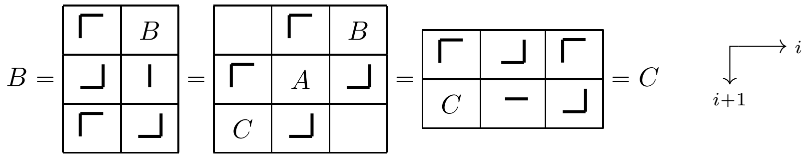
Definition 2.3.1. Let $\mathbf{C}$ be a cubical ω-category, and $i < n$ be integers. Let A, B be cells in $\mathbf{C}_n$ such that $\partial_i^\alpha A = \partial_{i+1}^\alpha B$ and $\partial_{i+1}^\alpha A = \partial_i^\alpha B$, for $\alpha = \pm$. If the following two equations are verified, we say that A is T_i*-invertible*, and that B is the T_i*-inverse* of A, and we denote B by $T_i A$:

$$\begin{array}{|c|c|}\hline \ulcorner & B \\ \hline A & \lrcorner \\ \hline \end{array} = \begin{array}{|c|c|}\hline \lrcorner & \ulcorner \\ \hline \end{array} \qquad \begin{tikzcd}[cramped] \cdot \arrow[r] \arrow[d] & i \\ i+1 & \end{tikzcd} \tag{2.5}$$

$$\begin{array}{|c|c|}\hline \ulcorner & A \\ \hline B & \lrcorner \\ \hline \end{array} = \begin{array}{|c|c|}\hline \lrcorner & \ulcorner \\ \hline \end{array} \qquad \begin{tikzcd}[cramped] \cdot \arrow[r] \arrow[d] & i \\ i+1 & \end{tikzcd} \tag{2.6}$$

In particular, we say that $A \in \mathbf{C}_{n+1}$ has a T_i-invertible shell if $\boldsymbol{\partial} A$ is T_i-invertible in $\square_n \mathbf{C}$.

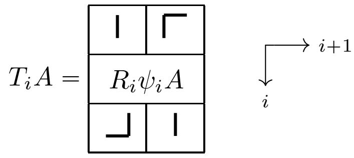
Remark 2.3.2. Note that T_iA is uniquely defined. Indeed, if B and C are both T_i-inverses of A, then evaluating the following square in two different ways shows that $B = C$:



The relationship between T_i and R_i-invertibility is given by the following Lemma.

Lemma 2.3.3. *Let* $\mathbf{C}$ *be a cubical* ω*-category, and* $A \in \mathbf{C}_n$ *be an* n*-cell, with* $n \geqslant 2$*. Then* A *is* T_i*-invertible (with* $i < n$*) if and only if* $\psi_i A$ *is* R_i*-invertible, and we have the equalities:*

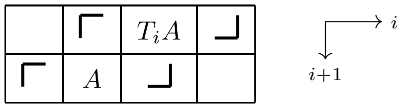
$$R_i\psi_i A = \psi_i T_i A \qquad (2.7)$$



(2.8)

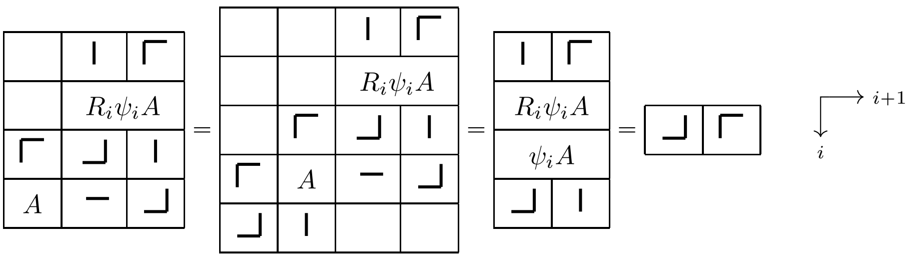
In particular, if A *is thin, then so is* T_iA*.*

Proof. Suppose first that A is T_i-invertible. Then the composite $\psi_i T_i A \star_i \psi_i A$ is equal to the following:



Using (2.5), we show that this composite is equal to $\epsilon_i \partial_i^+ \psi_i A$. We prove in the same way (using (2.6)), that $\psi_i A \star_i \psi_i T_i A = \epsilon_i \partial_i^- \psi_i A$, which shows that $\psi_i T_i A$ is the R_i-inverse of $\psi_i A$.

Suppose now that $\psi_i A$ is T_i-invertible. Then we have:



Finally, if A is thin, then so is $\psi_i A$, and so is $R_i\psi_i A$ by Proposition 2.2.2. Equation (2.8) then shows that T_iA is then thin. $\square$

Lemma 2.3.4. *Let* $\mathbf{C}$ *be a cubical* n*-category. Let* $1 \leqslant i < n$ *and* $A \in \square\mathbf{C}$*. Then* A *is* T_j*-invertible if and only if for all* $i \neq j, j+1$*,* A_i^α *is* T_{j_i}*-invertible, and:*

$$\partial_i^\alpha T_j A = \begin{cases} T_{j_i}\partial_i^\alpha A & i \neq j, j+1 \\ \partial_{j+1}^\alpha A & i = j, \\ \partial_j^\alpha A & i = j+1, \end{cases} \qquad (2.9)$$

In particular, if $\mathbf{C}$ *is a cubical* ω*-category, and a cell* $A \in \mathbf{C}_n$ *has a* T_i*-invertible shell, then* $\partial_j^\alpha A$ *is* T_{i_j}*-invertible for any* $j \neq i, i+1$*.*

Proof. Suppose first that $A \in \square\mathbf{C}$ is T_j-invertible, and let $i \neq j, j+1$. Then we have:

$$
\begin{aligned}
\partial_i^\alpha T_j A &= \begin{array}{|c|c|}\hline \partial_i^\alpha \epsilon_j \partial_{j+1}^- A & \partial_i^\alpha \Gamma_j^+ \partial_j^+ A \\ \hline \multicolumn{2}{|c|}{\partial_i^\alpha R_j \psi_j A} \\ \hline \partial_i^\alpha \Gamma_j^- \partial_j^- A & \partial_j^\alpha \epsilon_j \partial_{j+1}^+ A \\ \hline \end{array} \qquad \begin{tikzcd}[sep=small] \cdot \arrow[r, "(j+1)_i"] \arrow[d, "j_i"'] & {} \\ {} \end{tikzcd} \\
&= \begin{array}{|c|c|}\hline \epsilon_{j_i} \partial_{(j+1)_i}^- \partial_i^\alpha A & \Gamma_{j_i}^+ \partial_{j_i}^+ \partial_i^\alpha A \\ \hline \multicolumn{2}{|c|}{R_{j_i} \psi_j \partial_i^\alpha A} \\ \hline \Gamma_{j_i}^- \partial_{j_i}^- \partial_i^\alpha A & \epsilon_{j_i} \partial_{(j+1)_i}^+ \partial_i^\alpha A \\ \hline \end{array} \qquad \begin{tikzcd}[sep=small] \cdot \arrow[r, "(j+1)_i"] \arrow[d, "j_i"'] & {} \\ {} \end{tikzcd} \\
&= T_{j_i} \partial_i^\alpha A
\end{aligned}
$$

For $i = j$, we have:

$$
\begin{aligned}
\partial_i^- T_i A &= \partial_i^- \epsilon_i \partial_{i+1}^- A \star_i \partial_i^- \Gamma_i^+ \partial_i^+ A \\
&= \partial_{i+1}^- A \star_i \epsilon_i \partial_i^- \partial_i^+ A \\
&= \partial_{i+1}^- A
\end{aligned}
\qquad\qquad
\begin{aligned}
\partial_i^+ T_i A &= \partial_i^- \Gamma_i^- \partial_i^- A \star_i \partial_i^- \epsilon_i \partial_{i+1}^+ A \\
&= \epsilon_i \partial_i^+ \partial_i^- A \star_i \partial_{i+1}^+ A \\
&= \partial_{i+1}^+ A
\end{aligned}
$$

Finally for $i = j+1$:

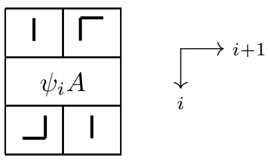
$$
\begin{aligned}
\partial_{i+1}^- T_i A &= \partial_{i+1}^- \epsilon_i \partial_{i+1}^- A \star_i \partial_{i+1}^- R_i \psi_i A \star_i \partial_{i+1}^- \Gamma_i^- \partial_i^- A \\
&= \epsilon_i \partial_i^- \partial_{i+1}^- A \star_i R_i \epsilon_i \partial_i^\alpha \partial_{i+1}^\alpha A \star_i \partial_i^- A \\
&= \epsilon_i \partial_i^\alpha \partial_{i+1}^\alpha A \star_i \partial_i^- A = \partial_i^- A
\end{aligned}
$$

$$
\partial_{i+1}^+ T_i A = \partial_{i+1}^+ \Gamma_i^+ \partial_i^+ A \star_i \partial_{i+1}^+ R_i \psi_i A \star_i \partial_{i+1}^+ \epsilon_i \partial_{i+1}^+ A
$$

Reciprocally suppose that for all $i \neq j, j+1$, A_j^α is T_{j_i}-invertible. Then let $B_i^\alpha = T_{j_i} A_i^\alpha$ if $i \neq j, j+1$, $B_j^\alpha = A_{j+1}^\alpha$ and $B_{j+1}^\alpha = A_j^\alpha$. Then B is an element of $\square\mathbf{C}$, and we verify that it is the T_i-inverse of A. ☐

Proposition 2.3.5. *Let $\mathbf{C}$ be a cubical ω-category, and $A \in \mathbf{C}_n$, with $n \geqslant 2$. Then A is T_i-invertible if and only if A is invertible and has a T_i-invertible shell.*

Proof. Suppose A is T_i-invertible. Then $\psi_i A$ is R_i-invertible, and therefore it is invertible. Recall from [1] that A is equal to the following composite:



All the cells in this composite are invertible, and invertible cells are closed under composition (Lemma 2.2.3), therefore A is invertible. Moreover since $\psi_i A$ is R_i-invertible, it has an R_i-invertible shell. In particular, for $j \neq i, i+1$, we have that $\partial_j^\alpha A = \partial_j^\alpha \psi_i A = \psi_{i_j} \partial_j^\alpha A$ is R_{i_j}-invertible. By Lemma 2.3.3, $\partial_j^\alpha A$ is T_{i_j}-invertible. So finally A has a T_j-invertible shell.

Suppose now that A is invertible and has a T_i-invertible shell. By application of Lemma 2.3.3 in $\square\mathbf{C}_n$, $\psi_i A$ is invertible and has an R_i-invertible shell. So $\psi_i A$ is R_i-invertible, and A is T_i-invertible by Lemma 2.3.3. ☐

Proposition 2.3.6. *Let $\mathbf{C}$ be a cubical ω-category.*

- *Let $A \in \mathbf{C}_n$. For all $1 \leqslant j \leqslant n+1$, $\epsilon_j A$ is T_j and T_{j-1}-invertible and:*

$$T_j \epsilon_j A = \epsilon_{j+1} A \qquad T_{j-1} \epsilon_j A = \epsilon_{j-1} A \tag{2.10}$$

  Moreover if A is T_i-invertible (for $i \neq j-1$), then $\epsilon_j A$ is T_{i^j}-invertible, and:

$$T_{i^j} \epsilon_j A = \epsilon_j T_i A \tag{2.11}$$

- *Let $A \in \mathbf{C}_n$. For all $1 \leqslant j \leqslant n$, $\Gamma_j^\alpha A$ is T_j-invertible, and*

$$T_j \Gamma_j^\alpha A = \Gamma_j^\alpha A \tag{2.12}$$

  Moreover, if A is T_i-invertible, then $\Gamma_j^\alpha A$ is T_{i^j}-invertible, and:

$$T_{i^j} \Gamma_j^\alpha A = \Gamma_j^\alpha T_i A \tag{2.13}$$

  Finally if A is T_i-invertible, then $\Gamma_{i+1}^\alpha A$ (resp. $\Gamma_i^\alpha A$) is T_i-invertible (resp. T_{i+1}-invertible) and $\Gamma_i^\alpha T_i A$ (resp. $\Gamma_{i+1}^\alpha T_i A$) is T_{i+1}-invertible (resp. T_i-invertible), and:

$$T_{i+1} \Gamma_i^\alpha T_i A = T_i \Gamma_{i+1}^\alpha A \qquad T_i \Gamma_{i+1}^\alpha T_i A = T_{i+1} \Gamma_i^\alpha A \tag{2.14}$$

- *Let $A, B \in \mathbf{C}_n$. If A and B are T_i-invertible, then $A \star_j B$ is T_i-invertible, and:*

$$T_i(A \star_j B) = \begin{cases} (T_i A) \star_{i+1} (T_i B) & j = i, \\ (T_i A) \star_i (T_i B) & j = i+1, \\ (T_i A) \star_j (T_i B) & \textit{otherwise.} \end{cases} \tag{2.15}$$

Proof. For the first seven equations, notice that both sides of the equations are thin by Lemma 2.3.3. Therefore by Theorem 1.2.5, it is enough to check that their shells are equal.

For the last one, we return to the definition of T_i-invertibility. $\square$

3 Relationship of cubical (ω, p)-categories with other structures

In Section 3.1, we collect the results of Section 2 to give a series of equivalent characterisation of the invertibility in a cubical ω-category of all cells of dimension n (Proposition 3.1.2). From that we then deduce the equivalence between globular and cubical (ω, p)-categories (Theorem 3.1.3).

In Section 3.2, we generalise the adjunctions between globular ω-groupoids and chain complexes and the one between globular ω-categories and ADCs from [29]. To do so we introduce the notion of (ω, p)-ADCs, such that (ω, ω)-ADCs are just ADCs, and $(\omega, 0)$-ADCs coincide with augmented chain complexes.

3.1 Cubical and globular (ω, p)-categories

In this Section we start by defining the notion of cubical (ω, p)-categories. In Proposition 3.1.2 we give various equivalent characterisations of those using the result from Section 2. As a result, we show Theorem 3.1.3 that the equivalence between globular and cubical ω-category induces equivalences between globular and cubical (ω, p)-categories. Finally in Corollary 3.1.4 we give a simple characterisation of the notions of cubical $(\omega, 0)$ and $(\omega, 1)$-categories.

Definition 3.1.1. Let $\mathbf{C}$ be a cubical ω-category, and p a natural number. We say that $\mathbf{C}$ is a *cubical* (ω, p)*-category* if any n-cell is invertible, for $n > p$. We denote by $(\omega, p)\text{-}\mathbf{CubCat}$ the full subcategory of $\omega\text{-}\mathbf{CubCat}$ spanned by cubical (ω, p)-categories.

Proposition 3.1.2. *Let $\mathbf{C}$ be a cubical ω-category, and fix $n > 0$. The following five properties are equivalent:*

1. *Any n-cell in $\mathbf{C}_n$ is invertible.*
2. *For all $1 \leqslant i \leqslant n$, any n-cell in $\mathbf{C}_n$ with an R_i-invertible shell is R_i-invertible.*
3. *Any n-cell in $\mathbf{C}_n$ with an R_1-invertible shell is R_1-invertible.*
4. *Any n-cell $A \in \mathbf{C}_n$ such that for all $j \neq 1$, $\partial_j^\alpha A \in \mathbf{Im}\,\epsilon_1$ is R_1-invertible.*
5. *Any n-cell in $\Phi_n(\mathbf{C}_n)$ is R_1-invertible.*

Moreover, if $n > 1$, then all the previous properties are also equivalent to the following:

6. *For all $1 \leqslant i < n$, any n-cell in $\mathbf{C}_n$ with a T_i-invertible shell is T_i-invertible*
7. *Any n-cell in $\mathbf{C}_n$ with a T_1-invertible shell is T_1-invertible.*

Proof. *(1)* $\Rightarrow$ *(2)* holds by Proposition 2.2.2, *(2)* $\Rightarrow$ *(3)* is clear, and *(3)* $\Rightarrow$ *(4)* holds because if $A \in \mathbf{C}_n$ satisfies $\partial_j^\alpha A \in \mathbf{Im}\,\epsilon_1$, then its shell is R_1-invertible. Also, *(4)* $\Rightarrow$ *(5)* holds because for any $A \in \Phi_n(\mathbf{C}_n)$, $\partial_j^\alpha A \in \mathbf{Im}\,\epsilon_1$ for all $j \neq 1$. Let us finally show that *(5)* $\Rightarrow$ *(1)*. From Lemmas 2.1.7 and 2.1.8, for any $i < n$, a cell $A \in \mathbf{C}_n$ with an R_1-invertible shell is R_1-invertible if and only if $\psi_i A$. Iterating this result, we get that for all $A \in \mathbf{C}_n$ $\psi_1 \ldots \psi_{n-1} A$ is R_1-invertible if and only if $\Phi\psi_1 \ldots \psi_{n-1} A$ is. Since $\Phi\psi_i = \Phi$ for all $i < n$, A is invertible if and only if ΦA is R_1-invertible.

Suppose now $n > 1$. Then *(1)* $\Rightarrow$ *(6)* by Proposition 2.3.5, and clearly *(6)* $\Rightarrow$ *(7)*. Suppose now that any n-cell with a T_1-invertible shell is T_1-invertible, and let us show that *(4)* holds. Let $A \in \mathbf{C}_n$ such that $\partial_j^\alpha A \in \mathbf{Im}\,\epsilon_1$ for all $j \neq 1$ is R_1-invertible. Then A has a T_1-invertible shell, and is therefore T_1-invertible by hypothesis. So A is invertible, and since it has an R_1-invertible shell, it is R_1-invertible. ◻

Theorem 3.1.3. *The functors λ and γ restrict to an equivalence of categories:*

$$\begin{tikzcd} (\omega, p)\text{-}\mathbf{Cat} \arrow[rr, bend right, "\gamma"'] \arrow[rr, phantom, "\cong"] & & (\omega, p)\text{-}\mathbf{CubCat} \arrow[ll, bend right, "\lambda"'] \end{tikzcd}$$

Proof. Let $\mathbf{C}$ be a cubical (ω, p)-category. The globular ω-category $\gamma\mathbf{C}$ is a globular (ω, p)-category if and only if, for all $n > p$, every cell in $\Phi_n(\mathbf{C}_n)$ is R_1-invertible. By Proposition 3.1.2, this is equivalent to $\mathbf{C}$ being a cubical (ω, p)-category. Since $(\omega, p)\text{-}\mathbf{Cat}$ and $(\omega, p)\text{-}\mathbf{CubCat}$ are replete full sub-categories respectively of $\omega\text{-}\mathbf{Cat}$ and $\omega\text{-}\mathbf{CubCat}$, this proves the result. ☐

Corollary 3.1.4. *Let $\mathbf{C}$ be a cubical ω-category. Then:*

- $\mathbf{C}$ *is a cubical ω-groupoid if and only if every n-cell of $\mathbf{C}$ is R_i-invertible for all $1 \leqslant i \leqslant n$.*
- $\mathbf{C}$ *is a cubical $(\omega, 1)$-category if and only if every n-cell is T_i-invertible, for all $1 \leqslant i < n$.*

Proof. If every n-cell of $\mathbf{C}_n$ is R_i-invertible then in particular every cell of $\mathbf{C}_n$ is invertible, and so $\mathbf{C}$ is a cubical ω-groupoid. Reciprocally, if $\mathbf{C}$ is a cubical ω-groupoid, we prove by induction on n that every cell is R_i-invertible. For $n = 1$, every 1-cell has an R_1-invertible shell, and so every cell is R_1-invertible. Suppose now the property true for all n-cells. Then any cell $A \in \mathbf{C}_{n+1}$ necessarily has a R_i-invertible shell by Lemma 2.1.2, and so the property holds for all $(n+1)$-cells.

The proof of the second point is similar, using the fact that any 2-cell in a cubical ω-category has a T_1-invertible shell. ☐

3.2 Augmented directed complexes and (ω, p)-categories

From [1] and [29], we have the following functors, where $\mathbf{ADC}$ is the category of augmented directed complexes.

$$\begin{tikzcd} \mathbf{ADC} \arrow[r, bend right, "N^{\mathbf{G}}"'] \arrow[r, phantom, "\perp"] & \omega\text{-}\mathbf{Cat} \arrow[l, bend right, "\mathcal{Z}^{\mathbf{G}}"'] \arrow[r, bend right, "\gamma"'] \arrow[r, phantom, "\cong"] & \omega\text{-}\mathbf{CubCat} \arrow[l, bend right, "\lambda"'] \end{tikzcd}$$

In this section we define cubical analogues to $N^{\mathbf{G}}$ and $\mathcal{Z}^{\mathbf{G}}$, and show that they induce an adjunction between $\mathbf{ADC}$ and $\omega\text{-}\mathbf{CubCat}$. Finally we show that all these functor can be restricted to the case of (ω, p)-categories, with a suitable notion of (ω, p)-$\mathbf{ADC}$.

Definition 3.2.1. An *augmented chain complex* K is a sequence of abelian groups K_n (for $n \geqslant 0$) together with applications $\mathrm{d} : K_{n+1} \to K_n$ for every $n \geqslant 0$ and an application $\mathrm{e} : K_0 \to \mathbb{Z}$ satisfying the equations:

$$\mathrm{d} \circ \mathrm{d} = 0 \qquad \mathrm{e} \circ \mathrm{d} = 0$$

A morphism of augmented chain complexes from $(K, \mathrm{d}, \mathrm{e}) \to (L, \mathrm{d}, \mathrm{e})$ is a family of morphisms $f_n : K_n \to L_n$ satisfying:

$$\mathrm{d} \circ f_{n+1} = f_n \circ \mathrm{d} \qquad \mathrm{e} = \mathrm{e} \circ f_0.$$

Definition 3.2.2. An *augmented directed chain complex* (or ADC for short) is an augmented chain complex K equipped with a submonoid K_n^* of K_n for any $n \geqslant 0$.

A morphism of ADCs $K \to L$ is a morphism of augmented chain complexes f satisfying $f(K_n^*) \subseteq L_n^*$. We denote by $\mathbf{ADC}$ the category of augmented directed chain complexes.

The following is a reformulation of Steiner [29]:

Proposition 3.2.3. *Let us fix $n \geqslant 0$, and let K the following ADC:*

$$K_k = \begin{cases} \mathbb{Z}[s_k, t_k] & k < n \\ \mathbb{Z}[x] & k = n \\ 0 & k > n \end{cases} \qquad K_k = \begin{cases} \mathbb{N}[s_k, t_k] & k < n \\ \mathbb{N}[x] & k = n \\ 0 & k > n \end{cases} \qquad \begin{cases} \mathrm{d}[x] = t_{n-1} - s_{n-1} \\ \mathrm{d}[s_{k+1}] = \mathrm{d}[t_{k+1}] = t_k - s_k \quad k \geqslant 0 \\ \mathrm{e}[s_0] = \mathrm{e}[t_0] = 1 \end{cases}$$

We denote this ADC by $n\text{-}\bullet^{\mathbf{ADC}}$. Equipped with morphisms $\check{\mathrm{s}}, \check{\mathrm{t}} : (n+1)\text{-}\bullet^{\mathbf{ADC}} \to n\text{-}\bullet^{\mathbf{ADC}}$, $\check{\mathrm{1}} : n\text{-}\bullet^{\mathbf{ADC}} \to (n+1)\text{-}\bullet^{\mathbf{ADC}}$ and $\check{\star}_i : n\text{-}\bullet^{\mathbf{ADC}} \bigsqcup_{i\text{-}\bullet^{\mathbf{ADC}}} n\text{-}\bullet^{\mathbf{ADC}}$, those form a co globular ω-category object in $\mathbf{ADC}$, and therefore they induce a functor $N^{\mathbf{G}} : \mathbf{ADC} \to \omega\text{-}\mathbf{Cat}$ defined by $(N^{\mathbf{G}}K)_n = \mathbf{ADC}(n\text{-}\bullet^{\mathbf{ADC}}, K)$

The category **ADC** is equipped with a tensor product defined as follows [29]:

Definition 3.2.4. Let K and L be ADCs. We define an object $K \otimes L$ in **ADC** as follows:

- For all $n \geqslant 0$, $(K \otimes L)_n = \bigoplus_{i+j=n} K_i \otimes L_j$.
- For all $n \geqslant 0$, $(K \otimes L)_n^*$ is the sub-monoid of $(K \otimes L)_n$ generated by the elements of the form $x \otimes y$, with $x \in K_i^*$ and $y \in L_{n-i}^*$.
- For all $x \in K_i$ and $y \in L_{n-i}$, $\mathrm{d}[x \otimes y] = \mathrm{d}[x] \otimes y + (-1)^i x \otimes \mathrm{d}[y]$.
- For all $x \in K_0$ and $y \in L_0$, $\mathrm{e}[x \otimes y] = \mathrm{e}[x]\,\mathrm{e}[y]$.

Proposition 3.2.5. *Let $\mathcal{C}$ be a globular ω-category. Following Steiner [29], we define an ADC $K = \mathcal{Z}^{\mathbf{G}}\mathcal{C}$ as follows:*

- *For all $n \in \mathbb{N}$, K_n is the quotient of the group $\mathbb{Z}[\mathcal{C}_n]$ by the relation $[A \bullet_k B] = [A] + [B]$.*
- *For all $n \in \mathbb{N}$, K_n^* is the image of $\mathbb{N}[\mathcal{C}_n]$ in K_n.*
- *For all $A \in \mathcal{C}_n$, $\mathrm{d}[A] = [\mathrm{s}(A)] - [\mathrm{t}(A)]$.*
- *For all $A \in \mathcal{C}_0$, $\mathrm{e}[A] = 1$.*

Proposition 3.2.6 ([29], Theorem 2.11)**.** *The functor $\mathcal{Z}^{\mathbf{G}}$ is left-adjoint to the functor $N^{\mathbf{G}}$.*

$$\begin{tikzcd} \mathbf{ADC} \arrow[rr, bend left, "\mathcal{Z}^{\mathbf{G}}"] \arrow[rr, phantom, "\perp"] & & \omega\text{-}\mathbf{Cat} \arrow[ll, bend left, "N^{\mathbf{G}}"] \end{tikzcd}$$

Definition 3.2.7. Let $n\text{-}\blacksquare^{\mathbf{ADC}}$ be the augmented directed complex $\mathcal{Z}^{\mathbf{G}}(n\text{-}\blacksquare^{\mathbf{G}})$. The applications $\check{\partial}_i^\alpha$, $\check{\epsilon}_i$, $\check{\Gamma}_i^\alpha$ and $\check{\star}_i$ still induce a structure of co-cubical ω-category object in **ADC** on the family $n\text{-}\blacksquare^{\mathbf{ADC}}$. Consequently, for any $K \in \mathbf{ADC}$ the family of sets $\mathbf{ADC}(n\text{-}\blacksquare^{\mathbf{ADC}}, K)$ is equipped with a structure of cubical ω-category. This defines a functor $N^{\mathbf{C}} : \mathbf{ADC} \to \omega\text{-}\mathbf{CubCat}$.

Let $\mathbf{C}$ be a cubical ω-category. We define an ADC $K = \mathcal{Z}^{\mathbf{C}}\mathbf{C}$ as follows:

- For all $n \in \mathbb{N}$, K_n is the quotient of $\mathbb{Z}[\mathbf{C}_n]$ by the relations $[A \star_k B] = [A] + [B]$ and $[\Gamma_i^\alpha A] = 0$.
- For all $n \in \mathbb{N}$, K_n^* is the image of $\mathbb{Z}[\mathbf{C}_n]$ is K_n.
- For all $A \in \mathbf{C}_n$,
$$\mathrm{d}[A] = \sum_{\substack{1 \leqslant i \leqslant n \\ \alpha = \pm}} \alpha(-1)^i [\partial_i^\alpha A]$$
- For all $A \in \mathbf{C}_0$, $\mathrm{e}[A] = 1$.

Proposition 3.2.8. *There are isomorphisms of functors:*

$$\mathcal{Z}^{\mathbf{C}} \approx \mathcal{Z}^{\mathbf{G}} \circ \gamma \qquad N^{\mathbf{C}} \approx \lambda \circ N^{\mathbf{G}}$$

As a result, we have the following diagram of equivalence and adjunctions between $\omega\text{-}\mathbf{Cat}$, $\omega\text{-}\mathbf{CubCat}$ and $\mathbf{ADC}$, where both triangles involving $\mathcal{Z}^{\mathbf{C}}$ and $\mathcal{Z}^{\mathbf{G}}$ and both triangles involving $N^{\mathbf{C}}$ and $N^{\mathbf{G}}$ commute up to isomorphism:

$$\begin{tikzcd} \omega\text{-}\mathbf{Cat} \arrow[rr, shift left, "\lambda"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift right, "\mathcal{Z}^{\mathbf{G}}"'] \arrow[dr, phantom, "\top"] & & \omega\text{-}\mathbf{CubCat} \arrow[ll, shift left, "\gamma"] \arrow[dl, shift left, "\mathcal{Z}^{\mathbf{C}}"] \arrow[dl, phantom, "\top"] \\ & \mathbf{ADC} \arrow[ul, shift right, "N^{\mathbf{G}}"'] \arrow[ur, shift left, "N^{\mathbf{C}}"] & \end{tikzcd}$$

Proof. Let K be an ADC. We have for all $n \geqslant 0$, using the adjunction between $N^{\mathbf{G}}$ and $\mathcal{Z}^{\mathbf{G}}$:

$$\begin{aligned}\lambda \circ N^{\mathbf{G}}(K)_n &= \omega\text{-}\mathbf{Cat}(n\text{-}\blacksquare^{\mathbf{G}}, N^{\mathbf{G}}K) \\ &\approx \mathbf{ADC}(\mathcal{Z}^{\mathbf{G}}(n\text{-}\blacksquare^{\mathbf{G}}), K) \\ &= \mathbf{ADC}(n\text{-}\blacksquare^{\mathbf{ADC}}, K) \\ &= (N^{\mathbf{C}}K)_n\end{aligned}$$

Moreover, because these equalities are functorial, they preserve the cubical ω-category structures on the families $\lambda \circ N^{\mathbf{G}}(K)_n$ and $(N^{\mathbf{C}}K)_n$. So finally we have the isomorphism $N^{\mathbf{C}} \approx \lambda \circ N^{\mathbf{G}}$.

Let now $\mathbf{C}$ be a cubical ω-category. For all $n \geqslant 0$, the group $\mathcal{Z}^{\mathbf{G}}(\gamma(\mathbf{C}))_n$ is the free abelian group generated by elements $[A]$, for $A \in \mathbf{Im}\,\Phi_n$, subject to the relations $[A \star_i B] = [A] + [B]$, for all $A, B \in \mathbf{Im}\,\Phi_n$. Let us show that for all $n \geqslant 0$, $\mathcal{Z}^{\mathbf{G}}(\gamma(\mathbf{C}))_n$ and $\mathcal{Z}^{\mathbf{C}}(\mathbf{C})_n$ are isomorphic.

First, the inclusion $\mathbf{Im}\,\Phi_n \to \mathbf{C}_n$ gives rise to an application $\mathbb{Z}[\mathbf{Im}\,\Phi_n] \to \mathcal{Z}^{\mathbf{C}}(\mathbf{C})_n$. Moreover this application respects the relations defining $Z^{\mathbf{G}}(\gamma(\mathbf{C}))_n$, so it induces a morphism $\iota : \mathcal{Z}^{\mathbf{G}}(\gamma(\mathbf{C}))_n \to \mathcal{Z}^{\mathbf{C}}(\mathbf{C})_n$.

For all $A \in \mathbf{C}_n$, we have in $\mathcal{Z}^{\mathbf{C}}(\mathbf{C})_n$: $[\psi_i A] = [\Gamma_i^+ \partial_{i+1}^- A] + [A] + [\Gamma_i^- \partial_{i+1}^+ A] = [A]$. By iterating this formula, we get that for all $A \in \mathbf{C}_n$, $[\Phi_n(A)] = [A]$. Hence ι is surjective. Let us now show that it is injective. Using the relation $[\Phi_n(A)] = [A]$, we get that $\mathcal{Z}^{\mathbf{C}}(\mathbf{C})_n$ is isomorphic to the free group generated by $[\mathbf{Im}\,\Phi_n]$, subject to the relations $[\Phi_n(A \star_i B)] = [\Phi_n(A)] + [\Phi_n(B)]$ for all $A, B \in \mathbf{C}_n$ and $[\Phi_n(\Gamma_i^\alpha A)] = 0$, for all $A \in \mathbf{C}_{n-1}$. Let us prove that these equalities already hold in $\mathcal{Z}^{\mathbf{G}}(\gamma(\mathbf{C}))_n$.

Let x be a thin cell in $\mathbf{C}_n$. Then $\Phi_n(x)$ is in the image of ϵ_1, and $\Phi_n(x) \star_1 \Phi_n(x) = \Phi_n(x)$, and so in $\mathcal{Z}^{\mathbf{G}}(\gamma(\mathbf{C}))_n$: $2 \cdot [\Phi_n(x)] = [\Phi_n(x)]$, and finally $[\Phi_n(x)] = 0$. In particular $[\Phi_n(\Gamma_i^\alpha A)] = 0$ in $\mathcal{Z}^{\mathbf{G}}(\gamma(\mathbf{C}))_n$. Let now A and B be i-composable n-cells. Following Proposition 6.8 from [1], $\Phi_n(A \star_i B)$ is a composite of cells of the form $\epsilon_1^{n-m}\Phi_m DA$ and $\epsilon_1^{n-m}\Phi_m DB$, where $0 \leqslant m \leqslant n$ is an integer, and D is a composite of length m of faces operations. Using the fact that $\epsilon_1^{n-m}\Phi_m = \Phi_n \epsilon_1^{n-m}$, we get that $\Phi_n(A \star_i B)$ is a composite of cells $\Phi_n(x)$, where x is thin, with the cells $\Phi_n(A)$ and $\Phi_n(B)$. So in $\mathcal{Z}^{\mathbf{G}}(\gamma(\mathbf{C}))_n$, $[\Phi_n(A \star_i B)] = k_1[\Phi_n(A)] + k_2[\Phi_n(B)]$ for some integers k_1 and k_2. Moreover, following Section 6 of [1], we verify that the cells $\Phi_n A$ and $\Phi_n B$ appear exactly once in this composition. As a result $[\Phi_n(A \star_i B)] = [\Phi_n(A)] + [\Phi_n(B)]$ in $\mathcal{Z}^{\mathbf{G}}(\gamma(\mathbf{C}))_n$. So $\mathcal{Z}^{\mathbf{G}}(\gamma(\mathbf{C}))_n$ and $\mathcal{Z}^{\mathbf{C}}(\mathbf{C})_n$ are isomorphic.

Let us denote respectively by $\mathrm{d}^{\mathbf{G}}$ and $\mathrm{d}^{\mathbf{C}}$ the boundary applications in $\mathcal{Z}^{\mathbf{G}}(\gamma(\mathbf{C}))$ and $\mathcal{Z}^{\mathbf{C}}(\mathbf{C})_n$. For $A \in \mathbf{Im}(\Phi_n)$, we have $\mathrm{d}^{\mathbf{G}}[A] = [\partial_1^- A] - [\partial_1^+ A]$, and $\mathrm{d}^{\mathbf{C}}[A] = \sum_{\substack{1 \leqslant i \leqslant n \\ \alpha = \pm}} \alpha(-1)^i [\partial_i^\alpha A]$. Since A is in $\mathbf{Im}\,\Phi_n$, for all $i \neq 1$, $\partial_i^\alpha A$ is thin. So $[\partial_i^\alpha A] = 0$, and $\mathrm{d}^{\mathbf{C}}[A] = [\partial_1^- A] - [\partial_1^+ A] = \mathrm{d}^{\mathbf{G}}[A]$. So ι induces an isomorphism of chain complexes between $\mathcal{Z}^{\mathbf{G}}(\gamma(\mathbf{C}))$ and $\mathcal{Z}^{\mathbf{C}}(\mathbf{C})$. Finally $Z^{\mathbf{G}}(\gamma(\mathbf{C}))_n^*$ and $\mathcal{Z}^{\mathbf{C}}(\mathbf{C})_n^*$ are the submonoids respectively generated by $\mathbf{Im}\,\Phi_n$ and $\mathbf{C}_n$ and $[A] = [\Phi_n(A)]$ in $\mathcal{Z}^{\mathbf{C}}(\mathbf{C})_n$, so $\mathcal{Z}^{\mathbf{G}}(\gamma(\mathbf{C}))$ and $\mathcal{Z}^{\mathbf{C}}(\mathbf{C})$ are isomorphic as ADCs. □

Definition 3.2.9. Let K be an ADC. We say that a cell $A \in K_n$ is *invertible* if $-A$ is in K_n^*.

We say that K is an (ω, p)-ADC if any $A \in K_n$ is invertible, for $n > p$. We denote by (ω, p)-**ADC** the category of (ω, p)-ADCs.

Remark 3.2.10. In particular, in an $(\omega, 0)$-ADC K, since every cell in K_n is invertible then $K_n = K_n^*$ (because $A = -(-A)$). So finally $(\omega, 0)$-**ADC** is just the category of chain complexes.

Proposition 3.2.11. *Let $\mathcal{C}$ be a globular ω-category, and $A \in \mathcal{C}_n$. If A is invertible, then so is $[A]$ in $\mathcal{Z}^{\mathbf{G}}(\mathcal{C})$, and $[A^{-1}] = -[A]$. In particular if $\mathcal{C}$ is an (ω, p)-category, then $\mathcal{Z}^{\mathbf{G}}\mathcal{C}$ is an (ω, p)-ADC.*

Let K be an ADC, and $A \in \mathbf{ADC}(n\text{-}\bullet^{\mathbf{ADC}}, K)$. If $A[x] \in K_n^*$ is invertible then so is A in $N^{\mathbf{G}}(K)$, and the inverse of A is given by:

$$B[x] = -A[x] \qquad \begin{cases} B[s_{n-1}] = A[t_{n-1}] \\ B[t_{n-1}] = A[s_{n-1}] \end{cases} \qquad \begin{cases} B[s_i] = A[s_i] & i < n-1 \\ B[t_i] = A[t_i] & i < n-1 \end{cases}$$

In particular if K is an (ω, p)-ADC then $N^{\mathbf{G}}K$ is a globular ω-category.

Proof. Let $\mathcal{C}$ be an ω-category, and $A \in \mathcal{C}_n$. If A is invertible, then there exists B such that $A \star_n B = 1_{\mathrm{s}(A)}$. Notice first that $[1_{\mathrm{s}(A)}] + [1_{\mathrm{s}(A)}] = [1_{\mathrm{s}(A)} \star_n 1_{\mathrm{s}(A)}] = [1_{\mathrm{s}(A)}]$, and therefore $[1_{\mathrm{s}(A)}] = 0$. So finally

$[A] + [B] = [A \star_n B] = 0$. Since both $[A]$ and $[B]$ are in $Z^{\mathbf{G}}(\mathcal{C})^*_n$, $[A]$ is invertible. If $\mathcal{C}$ is an (ω,p)-category, then for all $n > p$, $(\mathcal{Z}^{\mathbf{G}}\mathcal{C})^*_n$ is generated by invertible cells. Since invertible cells are closed under addition, $(\mathcal{Z}^{\mathbf{G}}\mathcal{C})^*_n$ is actually a group, and every cell in $(\mathcal{Z}^{\mathbf{G}}\mathcal{C})^*_n$ is invertible. So finally $\mathcal{Z}^{\mathbf{G}}\mathcal{C}$ is an (ω,p)-ADC.

Let now $A \in \mathbf{ADC}(n\text{-}\bullet^{\mathbf{ADC}}, K)$. Define B as the following morphism from $n\text{-}\bullet^{\mathbf{ADC}}$ to K:

$$B[x] = -A[x] \qquad \begin{cases} B[s_{n-1}] = A[t_{n-1}] \\ B[t_{n-1}] = A[s_{n-1}] \end{cases} \qquad \begin{cases} B[s_i] = A[s_i] & i < n-1 \\ B[t_i] = A[t_i] & i < n-1 \end{cases}$$

Note that since $A[x]$ is invertible, $-A[x]$ is in K^*_n, and so B is indeed a morphism of ADC. Moreover, A and B are $(n-1)$-composable, and $A \bullet_{n-1} B$ is given by:

$$(A \bullet_{n-1} B)[x] = A[x] - A[x] = 0 \qquad \begin{cases} (A \bullet_{n-1} B)[s_{n-1}] = A[s_{n-1}] \\ (A \bullet_{n-1} B)[t_{n-1}] = B[t_{n-1}] = A[s_{n-1}] \end{cases} \qquad \begin{cases} (A \bullet_{n-1} B)[s_i] = A[s_i] \\ (A \bullet_{n-1} B)[s_i] = A[s_i] \end{cases}$$

So $A \bullet_{n-1} B = 1_{\mathrm{s}(A)}$, and symmetrically $B \bullet_{n-1} A = 1_{\mathrm{t}(A)}$. The cell A is thus invertible. In particular if K is an (ω,p)-ADC, then for all $n > p$ and all $A \in \mathbf{ADC}(n\text{-}\bullet^{\mathbf{ADC}}, K)$, $A[x]$ is invertible and A is invertible. So every cell in $(N^{\mathbf{G}}K)_n$ is invertible, and finally $N^{\mathbf{G}}K$ is an (ω,p)-category. $\square$

Recall from [29] that $n\text{-}\blacksquare^{\mathbf{ADC}}_k$ is the free abelian group over the set $n\text{-}\blacksquare^{\mathbf{Set}}_k$ of sequences $s : \{1,\ldots,n\} \to \{(-),(\theta),(+)\}$ such that $|s^{-1}(\theta)| = k$. For any such s, and any $1 \leqslant i \leqslant n$ such that $s(i) \neq (\theta)$, we denote by $R_i s$ the sequence obtained by replacing $s(i)$ by $-s(i)$ in s. The following Proposition is the cubical analogue of the previous one.

Proposition 3.2.12. *Let $\mathbf{C}$ be an ω-category, and $A \in \mathbf{C}_n$. If A is R_i-invertible or T_i-invertible, then $[A]$ is invertible. In particular if $\mathbf{C}$ is a cubical (ω,p)-category, then $\mathcal{Z}^{\mathcal{C}}\mathbf{C}$ is an (ω,p)-ADC.*

Let K be an ADC, and let $A \in \mathbf{ADC}(n\text{-}\blacksquare^{\mathbf{ADC}}, K)$:

- *If for any $0 \leqslant k \leqslant n$, and any sequence $s \in n\text{-}\blacksquare^{\mathbf{Set}}_k$ such that $s(i) = (\theta)$, $A[s]$ is invertible (in K) then A is R_i-invertible, and R_iA is given by:*

$$R_iA[s] = \begin{cases} -A[s] & s(i) = (\theta) \\ A[R_is] & s(i) \neq (\theta) \end{cases}$$

- *If for any $0 \leqslant k \leqslant n$, and any sequence $s \in n\text{-}\blacksquare^{\mathbf{Set}}_k$ such that $s(i) = s(i+1) = (\theta)$, $A[s]$ is invertible, then A is T_i-invertible, and T_iA is given by:*

$$T_iA[s] = \begin{cases} -A[s] & s(i) = s(i+1) = (\theta) \\ A[s \circ \tau_i] & \textit{otherwise.} \end{cases}$$

In particular, if K is an (ω,p)-ADC, then $N^{\mathbf{C}}K$ is a cubical (ω,p)-category.

Proof. The proof is similar to that of the previous Proposition. $\square$

Theorem 3.2.13. *For all $p \in \mathbb{N} \cup \{\omega\}$, the categories $(\omega,p)\text{-}\mathbf{Cat}$, $(\omega,p)\text{-}\mathbf{CubCat}$ and (ω,p)-$\mathbf{ADC}$ are related by the following diagram of equivalence and adjunctions, where both triangles involving $\mathcal{Z}^{\mathbf{C}}$ and $\mathcal{Z}^{\mathbf{G}}$ and both triangles involving $N^{\mathbf{C}}$ and $N^{\mathbf{G}}$ commute up to isomorphism:*

$$\begin{tikzcd}
(\omega,p)\text{-}\mathbf{Cat} \arrow[rr, shift left, "\lambda"] \arrow[rr, phantom, "\cong"] \arrow[dr, shift right, "\mathcal{Z}^{\mathbf{G}}"'] & & (\omega,p)\text{-}\mathbf{CubCat} \arrow[ll, shift left, "\gamma"] \arrow[dl, shift left, "\mathcal{Z}^{\mathbf{C}}"] \\
& (\omega,p)\text{-}\mathbf{ADC} \arrow[ul, shift right, "N^{\mathbf{G}}"'] \arrow[ul, phantom, "\top"] \arrow[ur, shift left, "N^{\mathbf{C}}"] \arrow[ur, phantom, "\top"] &
\end{tikzcd}$$

Proof. We have already proven that the equivalence between $\omega\text{-}\mathbf{Cat}$ and $\omega\text{-}\mathbf{CubCat}$ could be restricted to (ω, p)-categories in Theorem 3.1.3, and by Propositions 3.2.11 and 3.2.12, so can the two adjunctions. Lastly, the commutations up to isomorphisms come from Proposition 3.2.8. $\square$

4 Permutations in cubical (ω, p)-categories

We apply our results from the previous Section to two different directions. First we show in Section 4.1 that the operations T_i induce a partial action of the symmetric group S_n on the n-cells of a cubical ω-category. To do this, we define a general notion of σ-invertibility, where $\sigma \in S_n$. In particular when σ is a transposition τ_i we recover the notion of T_i-invertibility of Section 2.3. In Section 4.2 we define the notions of lax and oplax transfors between cubical categories. We then define what it means for a tranfor to be pseudo using the notion of σ-invertibility defined previously and finally we show that the cubical ω-categories of pseudo lax and oplax transfors between two cubical ω-categories are isomorphic

4.1 Cubical $(\omega, 1)$-categories are symmetric

We start by defining a notion of u-invertibility, where u is a word over $T_1, \ldots, T_i$, and characterise the notion of u-invertibility in terms of plain invertibility, just as we have done previously for R_i and T_i-invertibility.

We then show how the notion of u-invertibility induces a notion of σ-invertibility, for $\sigma \in S_n$. The difficulty lies in the fact that, even if two words u and v over $T_1, \ldots, T_i$ correspond to the same permutations, the notions of u and v-invertibility do not necessarily coincide. We circumvent this difficulty by using a classical result about the symmetric group (see Theorem 4.1.11), which makes use of the notion of representative of minimal length of permutation.

Finally in Proposition 4.1.13 we extend the results concerning u-invertibility to σ-invertibility, with $\sigma \in S_n$.

Definition 4.1.1. Let $n \in \mathbb{N}$. We write $\mathbb{T}_n$ the free monoid on $n-1$ elements. We denote its generators by $T_1, \ldots, T_{n-1}$, and by $\mathrm{l} : \mathbb{T}_n \to \mathbb{N}$ the morphism of monoids that sends every T_i on 1. For $u \in \mathbb{T}_n$, we call $\mathrm{l}(u)$ the *length* of u.

Recall that S_n is a quotient of $\mathbb{T}_n$ using the relations:

$$T_i T_i = 1 \tag{4.1}$$

$$T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{4.2}$$

$$T_j T_i = T_j T_i \qquad |i-j| \geqslant 2 \tag{4.3}$$

We denote by $\bar{u}$ the image of an element $u \in \mathbb{T}_n$ in S_n, and $\tau_i = \bar{T}_i$. Using this projection, one defines a right-action of $\mathbb{T}_n$ on $\{1, \ldots, n\}$ by setting $k \cdot u := k \cdot \bar{u}$.

Let $\mathbf{C}$ be a cubical ω-category. For every $u \in \mathbb{T}_n$, we define a notion of u-invertible cell and a partial application $u \cdot \_ : \mathbf{C}_n \to \mathbf{C}_n$ defined on u-invertible cells as follows:

- Any n-cell of $\mathbf{C}_n$ is 1-invertible, and $1 \cdot A = A$.
- For any $u \in \mathbb{T}_n$ and $1 \leqslant i < n$, a cell $A \in \mathbf{C}_n$ is said to be $(T_i \cdot u)$-invertible if A is u-invertible and $u \cdot A$ is T_i-invertible. Moreover we set: $(T_i \cdot u) \cdot A := T_i(u \cdot A)$.

In particular we say that A has a u-invertible shell if $\boldsymbol{\partial} A$ is u-invertible in $\square_n \mathbf{C}$.

Proposition 4.1.2. *Let $\mathbf{C}$ be a cubical ω-category, and A be an n-cell in $\mathbf{C}$, with $n \geqslant 2$. Let $u \in \mathbb{T}_n$. Suppose $u \neq 1$. Then A is u-invertible if and only if A is invertible and has a u-invertible shell.*

Proof. We reason by induction on the length of u. If u is of length 1, there exists $1 \leqslant i < n$ such that $u = T_i$, and the result to prove becomes: *A is T_i-invertible if and only if A is invertible and has a T_i-invertible shell*, which is exactly Proposition 2.3.5.

Otherwise, write $u = T_i v$, with $v \neq 1$. Suppose A is u-invertible. Then by definition A is v-invertible, and $v \cdot A$ is T_i-invertible. By induction A is therefore invertible, and has a v-invertible shell. Moreover $v \cdot A$ is T_i-invertible, and hence has a T_i-invertible shell by Proposition 2.3.5. Since $\boldsymbol{\partial}(v \cdot A) = v \cdot \boldsymbol{\partial} A$, $\boldsymbol{\partial} A$ is v-invertible, and $v \cdot A$ is T_i-invertible. Therefore $\boldsymbol{\partial} A$ is u-invertible.

Reciprocally, suppose A is invertible, and has a $(T_i \cdot v)$-invertible shell. Then A has a v-invertible shell, and $v \cdot \boldsymbol{\partial} A$ is T_i-invertible. Since A is also invertible, by induction A is v-invertible, and since $\boldsymbol{\partial}(v \cdot A) = v \cdot \boldsymbol{\partial} A$, the cell $v \cdot A$ has a T_i-invertible shell. Moreover it is invertible, and so by Proposition 2.3.5, $v \cdot A$ is T_i-invertible, which means that A is u-invertible. $\square$

Definition 4.1.3. For $1 \leqslant i \leqslant n$, we define applications $\partial_i : \mathbb{T}_n \to \mathbb{T}_{n-1}$ as follows:

$$\partial_i 1 = 1 \qquad \partial_i T_j = \begin{cases} 1 & i = j, j+1 \\ T_{j_i} & i \neq j, j+1 \end{cases} \qquad \partial_i(u \cdot v) = \partial_i u \cdot \partial_{i \cdot u} v.$$

Note in particular that the applications ∂_i are *not* morphisms of monoids.

Lemma 4.1.4. *Let $u \in \mathbb{T}_n$. For all $1 \leqslant i \leqslant n$, and $1 \leqslant k \leqslant n$, we have:*

$$k \cdot \partial_i u = (k^i \cdot u)_{i \cdot u}$$

Proof. Note first the formula holds when u is 1 or a T_j. Finally, suppose the property holds for u and v. Then we have:

$$\begin{aligned} k \cdot \partial_i(u \cdot v) &= k \cdot \partial_i u \cdot \partial_{i \cdot u} v = (k^i \cdot u)_{i \cdot u} \cdot \partial_{i \cdot u} v \\ &= ((k^i \cdot u)^{i \cdot u}_{i \cdot u} \cdot v)_{i \cdot u \cdot v} = (k^i \cdot u \cdot v)_{i \cdot u \cdot v} \end{aligned}$$

□

Lemma 4.1.5. *Let $\mathbf{C}$ be a cubical n-category, $A \in (\Box\mathbf{C})_{n+1}$ and $u \in \mathbb{T}_{n+1}$. The cell A is u-invertible if and only if for all $j \leqslant n+1$, $A^\alpha_{j \cdot u}$ is $\partial_j u$-invertible, and:*

$$\partial^\alpha_j(u \cdot A) = \partial_j u \cdot \partial^\alpha_{j \cdot u} A$$

In particular, if $\mathbf{C}$ is a cubical ω-category, then $A \in \mathbf{C}_{n+1}$ has a u-invertible shell if and only if for all $j \leqslant n+1$, $\partial^\alpha_{j \cdot u} A$ is $\partial_j u$-invertible.

Proof. We reason by induction on the length of u. If u is of length 0, then $u = 1$ and for all j, $\partial_j u = 1$. Therefore both conditions are empty, and $(1 \cdot A)^\alpha_j = A^\alpha_j$.

Otherwise, write $u = T_i \cdot v$. Suppose that A is u-invertible. Then A is v-invertible, and $v \cdot A$ is T_i-invertible. Fix j and α. Then $\partial_j u = T_{i_j} \cdot \partial_{j \cdot T_i} v$. Let us show that $A^\alpha_{j \cdot u}$ is $\partial_j u$-invertible. We distinguish two cases:

- If $j = i$ (resp. $j = i+1$), then $\partial_j u = \partial_{i+1} v$ (resp. $\partial_j v$), and $j \cdot u = (i+1) \cdot v$ (resp. $i \cdot v$). By induction, $A^\alpha_{(i+1) \cdot v}$ (resp. $A^\alpha_{i \cdot v}$) is $\partial_{i+1} v$-invertible (resp. $\partial_i v$-invertible).
- Otherwise, then $\partial_j u = T_{i_j} \cdot \partial_j v$ and $j \cdot u = j \cdot v$. By induction hypothesis, $A^\alpha_{j \cdot v}$ is $\partial_j v$-invertible. Let us show that $\partial_j v \cdot A^\alpha_{j \cdot v}$ is T_{i_j}-invertible. First since A is $T_i \cdot v$-invertible, $v \cdot A$ is T_i-invertible, and so by Lemma 2.3.4, $\partial^\alpha_j(v \cdot A)$ is T_{i_j}-invertible. Finally by induction, $\partial^\alpha_j(v \cdot A) = \partial_j v \cdot A^\alpha_{j \cdot v}$.

Finally, using the induction property on v, we get:

$$(u \cdot A)^\alpha_j = (T_i \cdot v \cdot A)^\alpha_j = \begin{cases} (v \cdot A)^\alpha_{i+1} = \partial_{i+1} v \cdot A^\alpha_{(i+1) \cdot v} = \partial_i u \cdot A^\alpha_{i \cdot u} & j = i \\ (v \cdot A)^\alpha_i = \partial_i v \cdot A^\alpha_{i \cdot v} = \partial_{i+1} u \cdot A^\alpha_{(i+1) \cdot u} & j = i+1 \\ T_{i_j}(\partial_j v \cdot A)^\alpha_j = T_{i_j} \partial_j v \cdot A^\alpha_{j \cdot v} = \partial_j u \cdot A^\alpha_{j \cdot u} & j \neq i, i+1 \end{cases}$$

Suppose now that for all j, $A^\alpha_{j \cdot u}$ is $\partial_j u$-invertible. Let us show that A is u-invertible. First, let us prove that A is v-invertible. Indeed let $j \leqslant n$, and let us show that $A_{j \cdot v}$ is $\partial_j v$-invertible.

- If $j \neq i, i+1$, we have that $A^\alpha_{j \cdot u}$ is $\partial_j u$-invertible. Since $\partial_j u = T_{i_j} \partial_j v$, and $j \cdot u = j \cdot v$, this means that $A^\alpha_{j \cdot v}$ is $\partial_j v$-invertible (and $\partial_j v \cdot A^\alpha_{j \cdot v}$ is T_{i_j}-invertible.
- If $j = i$ (resp. $j = i+1$) then $\partial_{i+1} u = \partial_i v$ (resp. $\partial_i u = \partial_{i+1} v$) and $(i+1) \cdot u = i \cdot v$ (resp. $i \cdot u = (i+1) \cdot v$). So $A^\alpha_{j \cdot v}$ is $\partial_j v$-invertible.

Finally by induction, A is v-invertible. Let us show that $v \cdot A$ is T_i-invertible. Indeed, for $j \neq i, i+1$, $(v \cdot A)^\alpha_j = \partial_j v \cdot A^\alpha_{j \cdot v}$ is T_{i_j}-invertible, and so $v\dot{A}$ is T_i-invertible by Lemma 2.3.4. □

Lemma 4.1.6. *Let $\mathbf{C}$ be a cubical ω-category.*

- *If A is T_iT_i-invertible, then:*

$$T_iT_i \cdot A = A \tag{4.4}$$

- *A cell $A \in \mathbf{C}_n$ is $T_iT_{i+1}T_i$-invertible if and only if it is $T_{i+1}T_iT_{i+1}$-invertible, and*

$$T_iT_{i+1}T_iA = T_{i+1}T_iT_{i+1}A \tag{4.5}$$

- *Let $i, j < n$ such that $|i-j| \geqslant 2$. A cell $A \in \mathbf{C}_n$ is T_iT_j-invertible if and only if it is T_jT_i-invertible, and*

$$T_iT_j \cdot A = T_jT_i \cdot A \tag{4.6}$$

Proof. For the first one, notice that the axioms (2.5) and (2.6) are each other's symmetric, meaning that if B is the T_i-inverse of A, then A is the T_i-inverse of A. This means in particular that $T_iT_iA = A$.

For the second one, a cell $A \in \mathbf{C}_n$ is $T_iT_{i+1}T_i$-invertible if and only if it is invertible and $\boldsymbol{\partial} A$ is $T_iT_{i+1}T_i$-invertible, that is for all $j \leqslant n$, $\partial^\alpha_{j \cdot T_iT_{i+1}T_i} A$ is $\partial_j(T_iT_{i+1}T_i)$. Notice that:

$$\partial_j(T_iT_{i+1}T_i) = \begin{cases} T_{i_j}T_{i_j+1}T_{i_j} & j \neq i, i+1, i+2 \\ T_i & j = i, i+1, i+2 \end{cases} \qquad \partial_j(T_{i+1}T_iT_{i+1}) = \begin{cases} T_{i_j+1}T_{i_j}T_{i_j+1} & j \neq i, i+1, i+2 \\ T_i & j = i, i+1, i+2 \end{cases} \tag{4.7}$$

Therefore by induction on n, a cell is $T_iT_{i+1}T_i$-invertible if and only if it is $T_{i+1}T_iT_{i+1}$-invertible. Let A be such a cell. Let us show that $T_iT_{i+1}T_iA$ is the T_{i+1}-inverse of $T_iT_{i+1}A$. Indeed we have:

$$\begin{array}{|c|c|}\hline \Gamma^+_{i+1}T_i\partial^-_iA & T_iT_{i+1}A \\ \hline T_iT_{i+1}T_iA & \Gamma^-_{i+1}T_i\partial^+_{i+1}A \\ \hline \end{array} \quad \begin{array}{l} \lrcorner\!\!\to i+1 \\ \downarrow \\ i+2 \end{array} = T_iT_{i+1} \begin{array}{|c|c|}\hline \Gamma^+_i\partial^-_iA & A \\ \hline T_iA & \Gamma^-_i\partial^+_{i+1}A \\ \hline \end{array} \quad \begin{array}{l} \lrcorner\!\!\to i \\ \downarrow \\ i+1 \end{array}$$

$$\begin{aligned} &= T_iT_{i+1}(\Gamma^-_i\partial^+_{i+1}A \star_i \Gamma^+_i\partial^+_iA) \\ &= T_iT_{i+1}\Gamma^-_i\partial^+_{i+1}A \star_{i+1} T_iT_{i+1}\Gamma^+_i\partial^+_iA \\ &= \Gamma^-_{i+1}\partial^-_{i+2}T_iT_{i+1}A \star_{i+1} \Gamma^+_{i+1}\partial^+_{i+1}T_iT_{i+1}A \end{aligned}$$

The other axioms are verified in the same fashion. ☐

Definition 4.1.7. A symmetric cubical ω-category $\mathbf{C}$ is a cubical ω-category $\mathbf{C}$ equipped with (total) applications $T_i : \mathbf{C}_n \to \mathbf{C}_n$, for $1 \leqslant i \leqslant n-1$, satisfying the equalities (2.9) to (2.14) and (4.4) to (4.6).

Remark 4.1.8. Note that a symmetric cubical ω-category is close but not the same as the notion of symmetric cubical category defined by Grandis in [16]. A symmetric cubical category in the sense of Grandis would be a symmetric cubical ω-category (in the sense of 4.1.7, but without connections) object in the category **Cat**.

Proposition 4.1.9. *Let $\mathbf{C}$ be a cubical ω-category. The applications $A \mapsto T_iA$ induce a structure of symmetric cubical category on $\mathbf{C}$.*

Proof. The fact that the applications T_i are total is a consequence of Corollary 3.1.4, and the equations they verify are a consequence of Proposition 2.3.6 and Lemma 4.1.6. ☐

We now make explicit the (partial) action of the symmetric groups on the n-cells of a cubical category. To do so, we rely on Theorem 4.1.11, a classical result about the symmetric group.

Definition 4.1.10. For $u \in S_n$, we define the *length* of u as the integer $\mathrm{l}(u) = \min\{\mathrm{l}(v) | v \in \mathbb{T}_n \text{ and } \bar{v} = u\}$. A *representative of minimal length* of u in $\mathbb{T}_n$ is an element $v \in \mathbb{T}_n$ such that $\bar{v} = u$ and $\mathrm{l}(v) = \mathrm{l}(u)$.

Theorem 4.1.11. *Let $u, v \in \mathbb{T}_n$. If u and v are two representative of minimal length of a same permutation σ, then $u \equiv v$, where $\equiv$ is the congruence on $\mathbb{T}_n$ generated by (4.2) and (4.3).*

Definition 4.1.12. Let $\mathbf{C}$ be a cubical ω-category. For every $A \in \mathbf{C}_n$ and $\sigma \in S_n$, we say that A is σ-invertible if there exists a representative of minimal length u of σ such that A is u-invertible, and we define $\sigma \cdot A := u \cdot A$. By Lemma 4.1.6 and Theorem 4.1.11, this is independent from the choice of a minimal representative of σ.

Proposition 4.1.13. *The composites of the applications $\partial_i : \mathbb{T}_n \to \mathbb{T}_{n-1}$ with the projection $\mathbb{T}_{n-1} \twoheadrightarrow S_{n-1}$ are compatible with the relations (4.1) to (4.3). Hence they induce applications $\partial_i : S_n \to S_{n-1}$, satisfying:*

$$\partial_i 1 = 1 \qquad \partial_i \tau_j = \begin{cases} 1 & i = j, j+1 \\ \tau_{j_i} & i \neq j, j+1 \end{cases} \qquad \partial_i(\sigma \cdot \tau) = \partial_i \sigma \cdot \partial_{i\cdot\sigma}\tau.$$

Specifically, for $1 \leqslant i \leqslant n$ and $\sigma \in S_n$, $\partial_i \sigma$ is the (necessarily unique) permutation satisfying for all $1 \leqslant j \leqslant n-1$:

$$j \cdot \partial_i \sigma = (j^i \cdot \sigma)_{i\cdot\sigma} \tag{4.8}$$

Let $\mathbf{C}$ be a cubical n-category, and $\sigma \in S_n$. A cell $A \in (\square\mathbf{C})_{n+1}$ is σ-invertible if and only if for all $j \leqslant n$, $A^\alpha_{j\cdot\sigma}$ is $\partial_j\sigma$-invertible, and:

$$\partial^\alpha_j(\sigma \cdot A) = \partial_j \sigma \cdot \partial^\alpha_{j\cdot\sigma} A \tag{4.9}$$

Finally, let $\sigma \in S_n$. If $\sigma \neq 1$, then a cell $A \in \mathcal{C}_n$ is σ-invertible if and only if A is invertible and $\boldsymbol{\partial} A$ is σ-invertible.

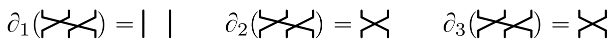
Proof. For the first point we simply verify the equalities as needed (note in particular that the compatibility of ∂_i with Equation (4.2) is a consequence of Equation (4.7).

The rest of the results is a consequence of Proposition 4.1.2, together with Lemma 4.1.4 and 4.1.5. ☐

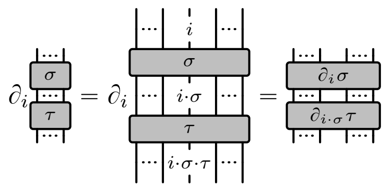
Remark 4.1.14. The operations ∂_i applied to a permutation σ correspond to deleting the i-th string in the string diagram representation of σ. For example, by definition we have:

$$\partial_1(\tau_1\tau_2) = (\partial_1\tau_1)\cdot(\partial_2\tau_2) = 1 \qquad \partial_2(\tau_1\tau_2) = (\partial_2\tau_1)\cdot(\partial_1\tau_2) = \tau_1 \qquad \partial_3(\tau_1\tau_2) = (\partial_3\tau_1)\cdot(\partial_3\tau_2) = \tau_1$$

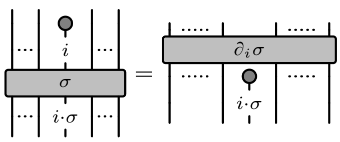
Which can be diagrammatically represented as:



More generally, the relation $\partial_i(\sigma\cdot\tau) = \partial_i\sigma\cdot\partial_{i\cdot\sigma}\tau$ corresponds to the diagram:



Finally, Equation (4.9) corresponds to the diagram:



Lemma 4.1.15. *Let $\mathbf{C}$ be a cubical ω-category, and $A \in \mathbf{C}_n$. If $\epsilon_i A$ is σ-invertible, then A is $\partial_{i\cdot\sigma^-}\sigma$-invertible and:*

$$\sigma \cdot \epsilon_i A = \epsilon_{i\cdot\sigma^-}(\partial_{i\cdot\sigma^-}\sigma \cdot A)$$

If Γ^α_i is σ-invertible then A is also $\partial_{i\cdot\sigma^-}\sigma$-invertible and if $(i+1)\cdot\sigma^- = i\cdot\sigma^- + 1$ we have:

$$\sigma \cdot \Gamma^\alpha_i A = \Gamma^\alpha_{i\cdot\sigma^-}(\partial_{i\cdot\sigma^-}\sigma \cdot A)$$

Proof. If $\epsilon_i A$ is σ-invertible, then $A = \partial^-_i \epsilon_i A$ is $\partial_{i\cdot\sigma^-}\sigma$ by Proposition 4.1.13.

To show the equality, we reason by induction on n. If $n = 0$ then $\sigma = 1$ and the result is verified. Otherwise, suppose $n > 0$. By Lemma 2.3.3, both sides of the equation are thin, and so they are equal if and only if their shells are equal. Note first that for $j = i\cdot\sigma^-$:

$$\partial^\alpha_j(\sigma\cdot\epsilon_i A) = \partial_j\sigma\cdot\partial^\alpha_i\epsilon_i A = \partial_j\sigma\cdot A = \partial^\alpha_j\epsilon_j(\partial_j\sigma\cdot A)$$

Now for $j \neq i \cdot \sigma^-$:

$$\partial_j^\alpha(\sigma \cdot \epsilon_i A) = \partial_j\sigma \cdot \partial_{j\cdot\sigma}^\alpha \epsilon_i A = \partial_j\sigma \cdot \epsilon_{i_{j\cdot\sigma}} \partial_{(j\cdot\sigma)_i}^\alpha A$$

Note that $\partial_j(\sigma \cdot \sigma^-) = \partial_j\sigma \cdot \partial_{j\cdot\sigma}\sigma^- = 1$, so $(\partial_j\sigma)^- = \partial_{j\cdot\sigma}\sigma^-$. So by proposition 4.1.13:

$$i_{j\cdot\sigma} \cdot (\partial_j\sigma)^- = (i_{j\cdot\sigma}^j \cdot \sigma^-)_{j\cdot\sigma\cdot\sigma^-} = (i \cdot \sigma^-)_j$$

So by induction hypothesis, we have $\partial_j^\alpha(\sigma \cdot \epsilon_i A) = \epsilon_{(i\cdot\sigma^-)_j}(\partial_{(i\cdot\sigma^-)_j}\partial_j\sigma \cdot \partial_{(j\cdot\sigma)_i}^\alpha A)$. On the other hand, note that $j_{i\cdot\sigma^-} \cdot \partial_{i\cdot\sigma^-}\sigma = (j_{i\cdot\sigma^-}^{i\cdot\sigma^-} \cdot \sigma)_{i\cdot\sigma^-\cdot\sigma} = (j \cdot \sigma)_i$. Applying this we get:

$$\partial_j^\alpha \epsilon_{i\cdot\sigma^-}(\partial_{i\cdot\sigma^-}\sigma \cdot A) = \epsilon_{(i\cdot\sigma^-)_j}\partial_{j_{i\cdot\sigma^-}}^\alpha(\partial_{i\cdot\sigma^-}\sigma \cdot A) = \epsilon_{(i\cdot\sigma^-)_j}(\partial_{j_{i\cdot\sigma^-}}\partial_{i\cdot\sigma^-}\sigma \cdot \partial_{(j\cdot\sigma)_i}^\alpha A).$$

Finally it remains to show that $\partial_{j_{i\cdot\sigma^-}}\partial_{i\cdot\sigma^-}\sigma = \partial_{(i\cdot\sigma^-)_j}\partial_j\sigma$. More generally, let us show that for any $i \neq j$, $\partial_{i_j}\partial_j\sigma = \partial_{j_i}\partial_i\sigma$. Indeed, for any k:

$$\partial_{i_j}\partial_j\sigma \cdot k = (((k^j)^{i_j} \cdot \sigma)_{i_j})_j = (k^{j,i} \cdot \sigma)_{i,j} \tag{4.10}$$

And this formula is symmetric in i and j by Lemma 1.1.3.

We now move on to the second equality. Once again if $\Gamma_i^\alpha A$ is σ-invertible, then $A = \partial_i^\alpha \Gamma_i^\alpha A$ is $\partial_{i\cdot\sigma^-}\sigma$-invertible by Proposition 4.1.13. We show the equality by induction on n. If $n = 1$, then the only permutation σ satisfying $(i+1)\cdot\sigma^- = i\cdot\sigma^- + 1$ is the identity, and the result is verified. Suppose now $n \geqslant 1$, and let $\sigma \in S_n$ such that $(i+1)\cdot\sigma^- = i\cdot\sigma^- + 1$. As previously, Lemma 2.3.3 show that both sides of the equation are thin, and so they are equal if and only if their shells are equal. Let us calculate their faces. Let $1 \leqslant j \leqslant n$ and $\beta = \pm$. We start by treating the case where $j = i \cdot \sigma^-$. For $\beta = \alpha$ we have:

$$\begin{aligned}\partial_j^\alpha(\sigma \cdot \Gamma_i^\alpha A) &= \partial_j\sigma \cdot \partial_{j\cdot\sigma}^\alpha \Gamma_i^\alpha A = \partial_j\sigma \cdot \partial_i^\alpha \Gamma_i^\alpha A \\ &= \partial_j\sigma \cdot A = \partial_j^\alpha \Gamma_j^\alpha(\partial_j\sigma \cdot A)\end{aligned}$$

Now for $\beta = -\alpha$. Note first that $j \cdot \partial_j\sigma = (j^j \cdot \sigma)_i = ((j+1)\cdot\sigma)_i = (i+1)_i = i$ (we here use the hypothesis on σ). Therefore $i \cdot (\partial_j\sigma)^- = j$, and:

$$\begin{aligned}\partial_j^{-\alpha}(\sigma \cdot \Gamma_i^\alpha A) &= \partial_j\sigma \cdot \partial_i^{-\alpha}\Gamma_i^\alpha A \\ &= \partial_j\sigma \cdot \epsilon_i \partial_i^{-\alpha} A \\ &= \epsilon_j(\partial_j\partial_j\sigma \cdot \partial_i^{-\alpha} A) \\ \partial_j^{-\alpha}\Gamma_j^\alpha(\partial_j\sigma \cdot A) &= \epsilon_j \partial_j^{-\alpha}(\partial_j\sigma \cdot A) \\ &= \epsilon_j(\partial_j\partial_j\sigma \cdot \partial_i^{-\alpha} A)\end{aligned}$$

The case where $j = i\cdot\sigma^- + 1$ is similar. We now study the general case where $\beta = \pm$ and $j \neq i\cdot\sigma^-, i\cdot\sigma^- + 1$:

$$\begin{aligned}\partial_j^\beta(\sigma \cdot \Gamma_i^\alpha A) &= \partial_j\sigma \cdot \partial_{j\cdot\sigma}^\beta \Gamma_i^\alpha A \\ &= \partial_j\sigma \cdot \Gamma_{i_{j\cdot\sigma}}^\alpha \partial_{(j\cdot\sigma)_i}^\beta A\end{aligned}$$

$$\begin{aligned}\partial_j^\beta \Gamma_{i\cdot\sigma^-}^\alpha(\partial_{i\cdot\sigma^-}\sigma \cdot A) &= \Gamma_{(i\cdot\sigma^-)_j}^\alpha \partial_{j_{i\cdot\sigma^-}}^\beta(\partial_{i\cdot\sigma^-}\sigma \cdot A) \\ &= \Gamma_{(i\cdot\sigma^-)_j}^\alpha(\partial_{j_{i\cdot\sigma^-}}\partial_{i\cdot\sigma^-}\sigma \cdot \partial_{j_{i\cdot\sigma^-}\cdot\partial_{i\cdot\sigma^-}\sigma}^\beta A)\end{aligned}$$

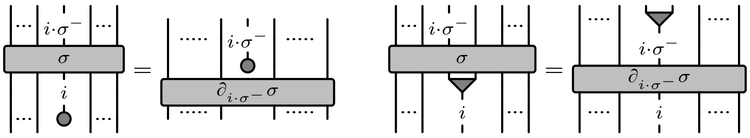
To conclude using the induction hypothesis, we need to show that $j_{i\cdot\sigma^-} \cdot \partial_{i\cdot\sigma^-}\sigma = (j \cdot \sigma)_i$, and that $i_{j\cdot\sigma} \cdot (\partial_j\sigma)^- = (i \cdot \sigma^-)_j$. And indeed we have:

$$j_{i\cdot\sigma^-} \cdot \partial_{i\cdot\sigma^-}\sigma = (j_{i\cdot\sigma^-}^{i\cdot\sigma^-} \cdot \sigma)_{i\cdot\sigma^-\cdot\sigma} = (j \cdot \sigma)_i$$

$$(i \cdot \sigma^-)_j \cdot \partial_j\sigma = ((i \cdot \sigma^-)_j^j \cdot \sigma)_{j\cdot\sigma} = i_{j\cdot\sigma}$$

□

Remark 4.1.16. Diagrammatically, the equations from Lemma 4.1.15 correspond to the following diagrams:



Remark 4.1.17. In this Section, we restricted ourselves to the T_i-inverses. However, all the results previous can be adapted to also consider the R_i-inverses. The action of the symmetric groups are then extended into an action of the Hyperoctahedral groups BC_n, which are the full groups of permutations of the hypercubes. A presentation of the group BC_n is given by the generators R_i (for $1 \leqslant i \leqslant n$ and T_i (for $1 \leqslant i < n$), subject to the relations:

$$T_iT_i = 1 \qquad T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \qquad T_jT_i = T_jT_i \quad |i-j| \geqslant 2$$
$$R_iR_i = 1 \qquad R_iR_j = R_jR_i \quad i \neq j$$
$$T_iR_i = R_{i+1}T_i \qquad T_iR_{i+1} = R_iT_i \qquad T_iR_j = R_jT_i \quad j \neq i, i+1$$

In particular the groups BC_n are Coxeter groups and they hence verify an analogue to Theorem 4.1.11, often called Matsumoto's Theorem [25].

4.2 Transfors between cubical ω-categories

Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, and $F, G : \mathcal{C} \to \mathcal{D}$ be functors. Recall that a natural transformation η from F to G is given by an application $\eta : \mathcal{C}_0 \to \mathcal{D}_1$ such that, for all $x \in \mathcal{C}_0$, $\mathrm{s}(\eta_x) = F(x)$, $\mathrm{t}(\eta_x) = G(x)$, and for all $f : x \to y \in \mathcal{C}_1$ the following diagram commutes:

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"'] & F(y) \arrow[d, "\eta_y"] \\ G(x) \arrow[r, "G(f)"'] & G(y) \end{tikzcd} \tag{4.11}$$

Natural transformations compose, and so for any categories $\mathcal{C}$ and $\mathcal{D}$ there is a category $\mathbf{Cat}(\mathcal{C}, \mathcal{D})$.

If $\mathcal{C}$ and $\mathcal{D}$ are two globular 2-categories, and $F, G : \mathcal{C} \to \mathcal{D}$ are two functors, then there are multiple ways to extend the notion of natural transformation. A *lax natural transformation* from F to G consists in applications $\eta : \mathcal{C}_0 \to \mathcal{D}_1$ and $\eta : \mathcal{C}_1 \to \mathcal{D}_2$, satisfying some compatibility conditions. In particular, for $f : x \to y \in \mathcal{C}_1$, the 2-cell $\eta_f \in \mathcal{D}_2$ is required to have the following source and target:

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"'] & F(y) \arrow[d, "\eta_y"] \arrow[dl, phantom, "\swarrow \eta_f"] \\ G(x) \arrow[r, "G(f)"'] & G(y) \end{tikzcd}$$

An *oplax natural transformation* requires the 2-cell η_f to be in the opposite direction. This leads to two different notions of the 2-category of functors between $\mathcal{C}$ and $\mathcal{D}$, where objects are functors from $\mathcal{C}$ to $\mathcal{D}$, 1-cells are lax (resp. oplax) natural transformations, and 2-cells are modifications. Modifications consist of an application $\mathcal{C}_0 \to \mathcal{D}_2$ satisfying some compatibility conditions. Notice that, if η is a lax natural transformation and η_f is invertible for all $f \in C_1$, then replacing η_f by its inverse yields an oplax natural transformation (and reciprocally when reversing the role of lax and oplax natural transformation). Such natural transformations are called *pseudo*.

More generally, if $\mathcal{C}$ and $\mathcal{D}$ are ω-categories, there are notions of lax and oplax k-transfors between them (following terminology by Crans [12]), consisting of applications $\mathcal{C}_n \to \mathcal{D}_{n+k}$, for all $n \geqslant 0$. In

particular, 0-transfors correspond to functors, and lax (resp. oplax) 1-transfors to lax (resp. oplax) natural transformations.

Similar constructions can be made in cubical ω-categories, and are recalled in Definition 4.2.1. This definition uses the notion of Crans-Grey tensor product between cubical ω-categories. One benefit of working in cubical categories is that this tensor product has a very natural expression in this setting, and so we are able to make explicit the conditions that transfors between cubical ω-categories have to satisfy. Next we define the notion of pseudo transfor, using the notion of σ-invertibility defined in Section 4.1. In Proposition 4.2.4 we give an alternative characterisation of pseudo transfors. Lastly we prove that the notions of pseudo lax and oplax transfors coincide in Proposition 4.2.6.

Definition 4.2.1. We exhibited in Section 1 a structure of *cubical ω-category* object in $\omega\text{-}\mathbf{Cat}^{\mathrm{op}}$ on the family $n\text{-}\blacksquare^{\mathbf{G}}$. Applying the functor λ, we obtain a structure of cubical ω-category object in $\omega\text{-}\mathbf{CubCat}^{\mathrm{op}}$ of the family $n\text{-}\blacksquare^{\mathbf{C}} := \lambda(n\text{-}\blacksquare^{\mathbf{G}})$.

Consequently, if $\mathbf{C}$ and $\mathbf{D}$ are cubical ω-categories, then both the families (of sets) $\mathbf{Lax}(\mathbf{C},\mathbf{D})_n = \omega\text{-}\mathbf{CubCat}(n\text{-}\blacksquare^{C} \otimes \mathbf{C}, \mathbf{D})$ and $\mathbf{OpLax}(\mathbf{C},\mathbf{D})_n = \omega\text{-}\mathbf{CubCat}(\mathbf{C} \otimes n\text{-}\blacksquare^{C}, \mathbf{D})$ come equipped with cubical ω-category structures (where we denote by $\otimes$ the monoidal product on $\omega\text{-}\mathbf{CubCat}$ as defined in [1]).

We call an element $F \in \mathbf{Lax}(\mathbf{C},\mathbf{D})_n$ (resp. $F \in \mathbf{OpLax}(\mathbf{C},\mathbf{D})_n$) an *lax n-transfor* (resp. an *oplax n-transfor*) from $\mathbf{C}$ to $\mathbf{D}$. Unfolding the definition of the monoidal product on $\omega\text{-}\mathbf{CubCat}$ as defined in [1], Section 10, a lax p-transfor (resp. oplax p-transfor) is a family of applications $F_n : \mathbf{C}_n \to \mathbf{D}_{n+p}$ satisfying the equations (4.12) to (4.15) (resp. (4.16) to (4.19)).

$$\partial^\alpha_{p+i} F_n(A) = F_{n-1}(\partial^\alpha_i A) \qquad (4.12) \qquad\qquad \partial^\alpha_i F_n(A) = F_{n-1}(\partial^\alpha_i A) \qquad (4.16)$$

$$F_n(\epsilon_i A) = \epsilon_{p+i} F_{n-1}(A) \qquad (4.13) \qquad\qquad F_n(\epsilon_i A) = \epsilon_i F_{n-1}(A) \qquad (4.17)$$

$$F_n(\Gamma^\alpha_i A) = \Gamma^\alpha_{p+i} F_{n-1}(A) \qquad (4.14) \qquad\qquad F_n(\Gamma^\alpha_i A) = \Gamma^\alpha_i F_{n-1}(A) \qquad (4.18)$$

$$F_n(A \star_i B) = F_n(A) \star_{p+i} F_n(B) \qquad (4.15) \qquad\qquad F_n(A \star_i B) = F_n(A) \star_i F_n(B) \qquad (4.19)$$

Moreover, the cubical ω-category structure on $\mathbf{Lax}(\mathbf{C},\mathbf{D})$ (resp. on $\mathbf{OpLax}(\mathbf{C},\mathbf{D})$) is given by the equations (4.20) to (4.23) (resp. (4.24) to (4.27)).

$$(\partial^\alpha_i F)_n(A) = \partial^\alpha_i(F_n(A)) \qquad (4.20) \qquad\qquad (\partial^\alpha_i F)_n(A) = \partial^\alpha_{n+i}(F_n(A)) \qquad (4.24)$$

$$(\epsilon_i F)_n(A) = \epsilon_i(F_n(A)) \qquad (4.21) \qquad\qquad (\epsilon_i F)_n(A) = \epsilon_{n+i}(F_n(A)) \qquad (4.25)$$

$$(\Gamma^\alpha_i F)_n(A) = \Gamma^\alpha_i(F_n(A)) \qquad (4.22) \qquad\qquad (\Gamma^\alpha_i F)_n(A) = \Gamma^\alpha_{n+i}(F_n(A)) \qquad (4.26)$$

$$(F \star_i G)_n(A) = F_n(A) \star_i G_n(A) \qquad (4.23) \qquad\qquad (F \star_i G)_n(A) = F_n(A) \star_{n+i} G_n(A) \qquad (4.27)$$

The following Proposition is a consequence of [1], Section 10.

Proposition 4.2.2. *Let $\mathbf{C}$ be a cubical ω-category. The functors $(\_ \otimes \mathbf{C})$ and $(\mathbf{C} \otimes \_)$ are respectively left-adjoint to the functors $\mathbf{Lax}(\mathbf{C}, \_)$ and $\mathbf{OpLax}(\mathbf{C}, \_)$. This implies that $\omega\text{-}\mathbf{CubCat}$ is a biclosed monoidal category.*

Definition 4.2.3. Let $n, m \geqslant 0$ be integers. We denote by $\rho_{n,m} \in S_{n+m}$ the following permutations:

$$i \cdot \rho_{n,m} := \begin{cases} i + n & i \leqslant n \\ i - n & i > n \end{cases}$$

Let $\mathbf{C}$ and $\mathbf{D}$ be cubical ω-categories. We say that a lax p-transfor $F : \mathbf{C} \to \mathbf{D}$ is *pseudo* if for all $A \in \mathbf{C}_n$, $F(A)$ is $\rho_{n,p}$-invertible. We say that an oplax p-transfor $F : \mathbf{C} \to \mathbf{D}$ is *pseudo* if for all $A \in \mathbf{C}_n$, $F(A)$ is $\rho_{p,n}$-invertible.

Proposition 4.2.4. *Let $\mathbf{C}$ and $\mathbf{D}$ be cubical ω-categories, and $F : \mathbf{C} \to \mathbf{D}$ a lax p-transfor (resp. an oplax p-transfor). Then F is pseudo if and only if:*

- *Either $p = 0$,*
- *Or $p > 0$, for all $n > 0$ and all $A \in \mathbf{C}_n$, $F(A)$ is invertible, and for all $1 \leqslant i \leqslant p$, $\partial_i^\alpha F$ is pseudo.*

Moreover, if F is pseudo, then so are $\Gamma_i^\alpha F$ ($1 \leqslant i \leqslant p$), $\epsilon_i F$ ($1 \leqslant i \leqslant p+1$) and, if G is a pseudo lax p-transfor (resp. pseudo oplax p-transfor) then $F \star_i G$ (if defined) is also pseudo, for $1 \leqslant i \leqslant p$.

Proof. Let us prove the result for pseudo lax p-transfors, the case of pseudo oplax p-transfors being similar. If $p = 0$, then for all n, $\rho_{n,p} = 1$. Since any cell in $\mathbf{D}$ is 1-invertible, any lax 0-transfor is pseudo.

Suppose now $p > 0$. Let $F \in \mathbf{Lax}(\mathbf{C}, \mathbf{D})_p$, and suppose F is pseudo. Let $n > 0$ and $A \in \mathbf{C}_n$. Then $\rho_{n,p} \neq 1$, and by Proposition 4.1.13, $F_n(A)$ is invertible. Moreover, for $1 \leqslant i \leqslant p$, $(\partial_i^\alpha F)_n(A) = \partial^\alpha_{(p+i)\cdot\rho_{n,p}}(F_n(A))$ is $\partial_{p+i}\rho_{n,p}$-invertible. Since $\partial_{p+i}\rho_{n,p} = \rho_{n,p-1}$, we just proved that for all $A \in \mathbf{C}_n$, $(\partial_i^\alpha F)_n(A)$ is $\rho_{n,p-1}$-invertible. So $\partial_i^\alpha F$ is pseudo.

Reciprocally, suppose that for all $n > 0$, $F_n(A)$ is invertible, and for all $1 \leqslant i \leqslant p$, $\partial_i^\alpha F$ is pseudo. We reason by induction on n to show that for all $A \in \mathbf{C}_n$, $F_n(A)$ is $\rho_{n,p}$-invertible. If $n = 0$, $\rho_{n,p} = 1$ and $F_n(A)$ is $\rho_{n,p}$-invertible. If $n \geqslant 1$, then $F(A)$ is invertible and for all $1 \leqslant i \leqslant p$, $\partial^\alpha_{(i+n)\cdot\rho_{n,p}}(F_n(A)) = (\partial_i^\alpha F)(A)$ is $\rho_{n,p-1}$-invertible. And for all $1 \leqslant i \leqslant n$, $\partial^\alpha_{i\cdot\rho_{n,p}}(F_n(A)) = F_{n-1}(\partial_i^\alpha A)$ is $\rho_{n-1,p}$-invertible by induction. In conclusion, $F_n(A)$ is invertible, and for all $1 \leqslant i \leqslant p+n$, $\partial_i^\alpha(F_n(A))$ is $\partial_i\rho_{n,p}$-invertible. By Proposition 4.1.13, $F_n(A)$ is $\rho_{n,p}$-invertible.

We reason by induction on p to show that, for any pseudo lax p-transfor. F, $\epsilon_i F$ and $\Gamma_i^\alpha F$ are pseudo. Let $A \in \mathbf{C}_n$. By equations (4.13) and (4.14), $(\epsilon_i F)(A)$ and $(\Gamma_i^\alpha F)(A)$ are thin cells, and so in particular are invertible. Moreover the cubical ω-category structure on $\mathbf{Lax}(\mathbf{C}, \mathbf{D})$ show that for all j, we have:

$$\partial_i^\alpha \epsilon_j F = \begin{cases} \epsilon_{j_i}\partial_{i_j}^\alpha F & i \neq j \\ F & i = j \end{cases} \qquad \partial_i^\alpha \Gamma_j^\beta F = \begin{cases} \Gamma_{j_i}^\beta \partial_{i_j}^\alpha F & i \neq j, j+1 \\ F & i = j, j+1 \text{ and } \alpha = \beta \\ \epsilon_j \partial_j^\alpha F & i = j, j+1 \text{ and } \alpha = -\beta \end{cases}$$

Using what we proved previously, $\partial_k^\alpha F$ is pseudo for all k, so by induction, $\partial_j^\alpha \epsilon_i F$ and $\partial_j^\beta \Gamma_i^\alpha F$ are always pseudo. Applying the criterion that we proved previously for a p-transfor to be pseudo, $\epsilon_i F$ and $\Gamma_i^\alpha F$ are pseudo.

Finally, we reason by induction on p to show that for any two pseudo lax p-transfors F and G, $F \star_i G$ is pseudo (if it is defined). Since any lax 0-transfor is pseudo, it is true if $p = 0$. Take now $p > 0$, and $A \in \mathbf{C}_n$, for some $n > 0$. Then $F(A)$ and $G(A)$ are invertible, and so is $(F \star_i G)_n(A) = F_n(A) \star_i G_n(A)$ by Lemma 2.2.3. Moreover, using the cubical ω-category structure on $\mathbf{Lax}(\mathbf{C}, \mathbf{D})$, we have:

$$\partial_i^\alpha(F \star_j G) = \begin{cases} \partial_i^\alpha F \star_{j_i} \partial_i^\alpha G & i \neq j \\ \partial_i^- F & i = j \text{ and } \alpha = - \\ \partial_i^+ G & i = j \text{ and } \alpha = + \end{cases}$$

So by the induction hypothesis, $\partial_j^\alpha(F \star_i G)$ is pseudo for all j. Therefore, $F \star_i G$ is pseudo. ☐

Definition 4.2.5. Let $\mathbf{C}$ and $\mathbf{D}$ be cubical ω-categories. We denote by $\mathbf{PsLax}(\mathbf{C}, \mathbf{D})$ (resp. $\mathbf{PsOpLax}(\mathbf{C}, \mathbf{D})$) the pseudo lax transfors (resp. the pseudo oplax transfors) from $\mathbf{C}$ to $\mathbf{D}$. By Proposition 4.2.4, $\mathbf{PsLax}(\mathbf{C}, \mathbf{D})$ and $\mathbf{PsOpLax}(\mathbf{C}, \mathbf{D})$ are cubical ω-categories.

Proposition 4.2.6. *For all cubical ω-categories $\mathbf{C}$ and $\mathbf{D}$, the cubical ω-categories $\mathbf{PsLax}(\mathbf{C}, \mathbf{D})$ and $\mathbf{PsOpLax}(\mathbf{C}, \mathbf{D})$ are isomorphic.*

Proof. Let $F \in \mathbf{PsLax}(\mathbf{C}, \mathbf{D})$, and define applications $G_n : \mathbf{C}_n \to \mathbf{D}_{n+p}$ as: $G_n(A) = \rho_{n,p} \cdot F_n(A)$. Let us show that G is an oplax p-transfor (using formulas from Lemma 4.1.15):

$$\begin{aligned} \partial_i^\alpha G_n(A) &= \partial_i^\alpha(\rho_{n,p} \cdot F_n(A)) = \partial_i \rho_{n,p} \cdot \partial^\alpha_{i\cdot\rho_{n,p}} F_n(A) \\ &= \rho_{n-1,p} \cdot \partial_{i+p} F_n(A) = \rho_{n-1,p} \cdot F_{n-1}(\partial_i^\alpha(A)) = G_{n-1}(\partial_i^\alpha(A)) \end{aligned}$$

$$\begin{aligned} G_n(\epsilon_i A) &= \rho_{n,p} \cdot F_n(\epsilon_i A) = \rho_{n,p} \cdot \epsilon_{p+i} F_{n-1}(A) \\ &= \epsilon_{(p+i)\cdot\rho_{p,n}}(\partial_{(p+i)\cdot\rho_{p,n}}\rho_{n,p} \cdot F_{n-1}(A)) \\ &= \epsilon_i(\partial_i \rho_{n,p} \cdot F_{n-1}(A)) = \epsilon_i(\rho_{n-1,p} \cdot F_{n-1}(A)) = \epsilon_i G_{n-1}(A) \end{aligned}$$

$$\begin{aligned} G_n(\Gamma_i^\alpha A) &= \rho_{n,p}\cdot F_n(\Gamma_i^\alpha A) = \rho_{n,p}\cdot \Gamma_{p+i}^\alpha F_{n-1}(A) \\ &= \Gamma^\alpha_{(p+i)\cdot\rho_{p,n}}(\partial_{(p+i)\cdot\rho_{p,n}}\rho_{n,p}\cdot F_{n-1}(A)) \\ &= \Gamma_i^\alpha(\partial_i\rho_{n,p}\cdot F_{n-1}(A)) = \Gamma_i^\alpha(\rho_{n-1,p}\cdot F_{n-1}(A)) = \Gamma_i^\alpha G_{n-1}(A) \end{aligned}$$

$$\begin{aligned} G_n(A\star_i B) &= \rho_{n,p}\cdot F_n(A\star_i B) = \rho_{n,p}\cdot(F_n(A)\star_{p+i} F_n(B)) \\ &= (\rho_{n,p}\cdot F_n(A))\star_{(p+i)\cdot\rho_{p,n}}(\rho_{n,p}\cdot F_n(B)) = G_n(A)\star_i G_n(B) \end{aligned}$$

We denote by $\mathbf{P}(F)$ this oplax p-transfor. Moreover for $A\in\mathbf{C}_n$ $\rho\cdot F(A) = \rho_{n,p}\cdot F(A)$ is $\rho_{p,n}$-invertible (with $\rho_{p,n}$-inverse A). So $\mathbf{P}(F)$ is actually pseudo. Let us show that $\mathbf{P}$ is functorial. Let $F\in\mathbf{PsLax}(\mathbf{C},\mathbf{D})_p$:

$$\begin{aligned} (\partial_i^\alpha(\mathbf{P}(F)))_n(A) &= \partial^\alpha_{n+i}((\mathbf{P}(F))_n(A)) = \partial^\alpha_{n+i}(\rho_{n,p}\cdot F(A)) \\ &= \partial_{n+i}\rho_{n,p}\cdot\partial^\alpha_{(n+i)\cdot\rho_{n,p}}F(A) \\ &= \rho_{n,p-1}\cdot\partial_i^\alpha F(A) = \mathbf{P}(\partial_i^\alpha F)(A) \end{aligned}$$

$$\begin{aligned} (\mathbf{P}(\Gamma_i^\alpha F))_n(A) &= \rho_{n,p}\cdot((\Gamma_i^\alpha F)_n(A)) = \rho_{n,p}\cdot\Gamma_i^\alpha(F_n(A)) \\ &= \Gamma^\alpha_{i\cdot\rho_{p,n}}(\partial_{i\cdot\rho_{p,n}}\rho_{n,p}\cdot F_n(A)) \\ &= \Gamma^\alpha_{n+i}(\partial_{p+i}\rho_{n,p}\cdot F_n(A)) = \Gamma^\alpha_{n+i}(\rho_{n,p-1}\cdot F_n(A)) = (\Gamma_i^\alpha(\mathbf{P}(F)))_n(A) \end{aligned}$$

$$\begin{aligned} (\mathbf{P}(\epsilon_i F))_n(A) &= \rho_{n,p}\cdot((\epsilon_i F)_n(A)) = \rho_{n,p}\cdot\epsilon_i(F_n(A)) \\ &= \epsilon_{i\cdot\rho_{p,n}}(\partial_{i\cdot\rho_{p,n}}\rho_{n,p}\cdot F_n(A)) \\ &= \epsilon_{n+i}(\partial_{p+i}\rho_{n,p}\cdot F_n(A)) = \epsilon_{n+i}(\rho_{n,p-1}\cdot F_n(A)) = (\epsilon_i(\mathbf{P}(F)))_n(A) \end{aligned}$$

$$\begin{aligned} (\mathbf{P}(F\star_i G))_n(A) &= \rho_{n,p}\cdot((F\star_i G)_n(A)) = \rho_{n,p}\cdot(F_n(A)\star_i G_n(A)) \\ &= (\rho_{n,p}\cdot F_n(A))\star_{i\cdot\rho_{p,n}}(\rho_{n,p}\cdot G_n(A)) \\ &= \mathbf{P}(F)_n(A)\star_i\mathbf{P}(G)_n(A) = (\mathbf{P}(F)\star_i\mathbf{P}(G))_n(A) \end{aligned}$$

So $\mathbf{P}$ is a functor from $\mathbf{PsLax}(\mathbf{C},\mathbf{D})$ to $\mathbf{PsOpLax}(\mathbf{C},\mathbf{D})$. Reciprocally, if F is a pseudo oplax p-transfor, we define a family of applications $\mathbf{R}(F)_n : \mathbf{C}_n \to \mathbf{D}_{n+p}$ by setting $\mathbf{R}(F)_n(A) = \rho_{p,n}\cdot F_n(A)$. As we did for $\mathbf{P}$, we show that $\mathbf{R}$ induces a functor from $\mathbf{PsOpLax}(\mathbf{C},\mathbf{D})$ to $\mathbf{PsLax}(\mathbf{C},\mathbf{D})$. Finally, since $\rho_{p,n}\cdot\rho_{n,p} = 1$, $\mathbf{P}$ and $\mathbf{R}$ are inverses of each other, and $\mathbf{PsLax}(\mathbf{C},\mathbf{D})$ is isomorphic to $\mathbf{PsOpLax}(\mathbf{C},\mathbf{D})$. ☐

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